1. Problem 5.3 from Griffiths

This problem concerns the method followed by J. J. Thomson during his work to experimentally measure the charge to mass ratio of the electron.

(a) Thomson passed a beam of cathode rays (electrons) through electric and magnetic fields. These fields are perpendicular to each other and also perpendicular with respect to the beam. Then he adjusted the electric field until there was no deflection of the beam. Express the speed of the particles in terms of $E$ and $B$.

(b) Thomson then turned off the electric field and measured the radius of curvature, $R$, of the beam. Write the charge to mass ratio of the electron in terms of $E$, $B$, and $R$.

(a) Velocity to Prevent Beam Deflection

The total force on the electrons in the presence of these fields is given by the Lorentz force law,

$$ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (1) $$

The geometry of the problem determines that the force due to the electric field is already directed oppositely from the magnetic force. Set the magnitudes of the electric and magnetic forces equal to determine the velocity at which they cancel out.

$$ q\vec{v} \times \vec{B} = q\vec{E} \quad (2) $$

$$ vB \sin \theta = E \quad \text{where } \theta = 90^\circ $$

$$ v = \frac{E}{B} $$
This is simply the drift velocity due to the $\vec{E} \times \vec{B}$ force. Recall that this drift velocity is given most generally as,

$$\vec{v}_{\vec{E} \times \vec{B}} = \frac{\vec{E} \times \vec{B}}{B^2}$$  \hspace{1cm} (3)

and in our problem $\vec{E} \perp \vec{B}$ so $\vec{v}_{\vec{E} \times \vec{B}}$ simplifies to the solution we just found for $v$.

(b) Charge to Mass Ratio

The radius of curvature for an electron beam in a magnetic field is given by,

$$R = \frac{vm}{qB} \hspace{1cm} \text{Eq. 5.3}$$  \hspace{1cm} (4)

The velocity has been determined from the steps in Part (a) so we can rearrange this equation to solve for the charge to mass ratio,

$$\frac{q}{m} = \frac{E}{RB^2}$$

[2.] Problem 5.8 from Griffiths

(a) Find the magnetic field at the center of a square loop. Let the loop carry a current of $I$ and let $R$ be the distance from any side to the center. Reference figure 5.22 in Griffiths.

(b) Find the magnetic field at the center of a regular polygon with $n$ sides. This loop has the same $I$ and $R$ dimension as the loop in Part (a)

(c) Show that the field you found in Part (b) reduces to the expected value for a circular loop when $n \to \infty$.

(a) $\vec{B}$ at Center of Square Loop

Using the right-hand rule it is seen that each side of the loop contributes an equal amount to the total magnetic field at the center. Figure 5.22 illustrates that this field will be directed out of the page. The total field is then given by four times the contribution due to any one of them.

$$\vec{B}_{\text{total}} = 4\vec{B}_{\text{side}}$$  \hspace{1cm} (5)

Griffiths has already solved for the magnetic field away from a segment of current carrying wire. This is a magnitude, we are able to determine the direction of the field using the right-hand rule since we know the current in the wire.

$$B = \frac{\mu_0 I}{4\pi s} (\sin \theta_2 - \sin \theta_1)$$  \hspace{1cm} \text{Eq. 5.35}  \hspace{1cm} (6)

In (6) the angles determine the starting and ending points of the wire segment. Example 5.5 in Griffiths derives this expression entirely.
Using (6) for one of the sides of our square loop we see that,

\[ \theta_1 = -45^\circ = -\frac{\pi}{4} \]

\[ \theta_2 = 45^\circ = \frac{\pi}{4} \]

Since \( \sin \theta \) is an odd function (i.e. \( \sin(-\theta) = -\sin \theta \)) and \( \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \),

\[ B_{\text{side}} = \frac{\mu_o I}{4\pi R} \left[ 2\frac{\sqrt{2}}{2} \right] = \frac{\sqrt{2}\mu_o I}{4\pi R} \]

The field at the center is four times this value and directed out of the page.

\[ B_{\text{total}} = \frac{\sqrt{2}\mu_o I}{\pi R} \quad \text{(7)} \]

(b) \( \vec{B} \) at Center of \( n \) Sided Loop

We want to find the contribution to the field at the center from any one side of the polygon and then multiply that value by \( n \) to get the total field. Beginning with (6) we say that any regularly shaped polygon will have a field at the center given by (since \( \theta_1 = -\theta_2 \)),

\[ B_{\text{center}} = n\frac{\mu_o I}{2\pi R} \sin \theta_1 \quad \text{(8)} \]

\( \theta_1 \) is half of the total angle swept out across any one of the sides of our \( n \) sided polygon. The entire polygon must have 360\(^\circ\) so we can write,

\[ \theta_1 = \frac{1}{2} \left( \frac{360^\circ}{n} \right) = \frac{2\pi}{2n} = \frac{\pi}{n} \quad \text{(9)} \]

Now put the expression for \( \theta_1 \) into (8),

\[ B_{\text{center}} = \frac{\mu_o I}{2\pi R} n \sin \left( \frac{\pi}{n} \right) \quad \text{(10)} \]

(c) Taking the limit \( n \to \infty \)

The magnetic field on the axis of a circular loop is given as Eq. 5.38 in Griffiths,

\[ B = \frac{\mu_o I}{2} \left( \frac{R^2}{(R^2 + z^2)^{3/2}} \right) \quad \text{(11)} \]

In our specific problem we only care about the center of the loop in the plane of the loop, i.e. at \( z = 0 \). Therefore, the solution we want to match is the one given by (11) for \( z = 0 \).

\[ B = \frac{\mu_o I}{2R} \quad \text{(12)} \]
We take the limit using (10), the general formula for the magnetic field at the center of an \( n \) sided polygon.

\[
B_{n \to \infty} = \lim_{n \to \infty} \frac{n\mu_0 I}{2\pi R} \sin \left( \frac{\pi}{n} \right)
\]

As \( n \) gets very large the term \( \pi/n \) gets very small. The small angle approximation (most commonly used while solving for the motion of a pendulum with small oscillations) says that the sine of a small angle is approximately equal to the angle itself.

\[
B_{n \to \infty} = \frac{n\mu_0 I \pi}{2\pi R n} = \frac{\mu_0 I}{2R}
\]

Thus matching the expected result.

[3.] Problem 5.10 from Griffiths

(a) Reference the square loop of figure 5.24(a) in Griffiths. Find the force on the square loop due to the infinite wire below it. Both wires carry current \( I \).

(b) Find the force on the triangular loop shown in figure 5.24(b) of Griffiths. Again, both loops carry a current \( I \).

(a) Square Loop

I define the x-axis to be parallel to the direction of the infinite wire. The y-axis is parallel to the length \( s \), and the z-axis therefore is coming out of the page. The magnetic field due to the infinite wire is coming out of the page in the region above the wire. The forces we are dealing with here are of the \( \vec{I} \times \vec{B} \) type. Using the right-hand rule we can show that the forces on the vertical lengths of wires (those perpendicular to the infinite wire) will cancel. Both of these forces point inward. The force on this entire loop is given by the sum of the forces on the two wires running parallel to the infinite wire. Each force is given by,

\[
\vec{F} = \int I(d\vec{l} \times \vec{B})
\]

The magnetic field a distance \( s \) from the infinite wire is known,

\[
B = \frac{\mu_0 I}{2\pi s} \quad \text{Eq. 5.36 (14)}
\]

and in this problem it is directed along the z-axis in the region of the square loop. The force on the lower wire is,

\[
\vec{F}_{\text{lower}} = \int I(d\vec{l} \times \frac{\mu_0 I}{2\pi s} \hat{z})
\]

\[
= \frac{\mu_0 I^2}{2\pi s} \int_0^a dx \hat{x} \times \hat{z}
\]

\[
= \frac{\mu_0 aI^2}{2\pi s} \hat{y}
\]
Notice that I prefer to designate the direction of the line integral by taking the appropriate limits. You could also have switched the limits and then written the $d\vec{l} = -dx\hat{x}$. The force on the upper wire is given by,

$$\vec{F}_{\text{upper}} = \int_0^a Idx\hat{x} \times \frac{\mu_0I}{2\pi(s+a)}\hat{z}$$

$$= -\frac{\mu_0aI^2}{2\pi(s+a)}\hat{y}$$

These forces are directed opposite each other just as the right-hand rule indicated. The total force on the square loop is,

$$\vec{F} = \vec{F}_{\text{lower}} + \vec{F}_{\text{upper}} = \frac{\mu_0aI^2}{2\pi} \left( \frac{1}{s} - \frac{1}{s+a} \right) \hat{y}$$

(15)

Since $1/s > 1/(s+a)$ it is seen that the net force is away from the infinite wire.

(b) Triangular Loop

![Diagram of a triangular loop with magnetic field $\vec{B}_\infty$.]

Again the method is to find the force on each section of the loop. From Part (a) we already know the force on the section that is parallel to the infinite wire.

$$\vec{F}_{\text{bottom}} = \frac{\mu_0aI^2}{2\pi s} \hat{y}$$

(16)

using the same coordinates as Part (a). The other two sections of wire are slightly more difficult. One method is to use the right-hand rule and state that the forces along the $x$ direction for these wires cancel.
The wire on the left side has an x-component of the magnetic force that is directed oppositely the x-component of the other. Here I will write out the math and we will see that this statement is true. Note that the value $h$ in the figure is $\frac{a\sqrt{3}}{2}$, known from the geometry of the triangle.

These remaining sections of wire do not see a constant magnetic field due to the infinite wire. The field now depends on the y-coordinate and is given by,

$$\vec{B}_{\infty \text{wire}} = \frac{\mu_0 I}{2\pi y} \hat{z}$$

(17)

The force on the left wire is given by,

$$\vec{F}_{\text{left}} = \int I d\vec{l} \times \frac{\mu_0 I}{2\pi y} \hat{z}$$

$$= \frac{\mu_0 I^2}{2\pi} \int \left[(dx\hat{x} + dy\hat{y}) \times \frac{\hat{z}}{y}\right]$$

$$= \frac{\mu_0 I^2}{2\pi} \left[\int_0^{a/2} \left(dx\hat{x} \times \frac{\hat{z}}{y}\right) + \int_s^{s + \frac{a\sqrt{3}}{2}} \left(dy\hat{x} \times \frac{\hat{z}}{y}\right)\right]$$

$$= \frac{\mu_0 I^2}{2\pi} \left[\int_0^{a/2} \left(-\frac{\hat{y}}{y} dx\right) + \int_s^{s + \frac{a\sqrt{3}}{2}} \left(\frac{\hat{x}}{y} dy\right)\right]$$

The $dx$ and $dy$ terms separate the integrals into a sum instead of a product. The integral over $x$ is not as simple as it may seem because the value of $y$ depends on $x$ and we must rewrite the integral to take this into account. This section of wire is a straight line and $y$ can therefore be expressed as a simple function of $x$ in the form $y = mx + b$.

Initial point of line $\quad x = 0 \quad y = s \quad \rightarrow b = s$

Final point of line $\quad x = \frac{a}{2} \quad y = s + \frac{a\sqrt{3}}{2}$

$\therefore \quad m = \sqrt{3}$

For the left wire,

$$y = \sqrt{3} \; x + s$$

(18)

Now both integrals can be solved.

$$\vec{F}_{\text{left}} = \frac{\mu_0 I^2}{2\pi} \left[\int_0^{a/2} \frac{-\hat{y}}{\sqrt{3}} dx + \int_s^{s + \frac{a\sqrt{3}}{2}} \frac{\hat{x}}{y} dy\right]$$

Use substitution $\rightarrow \quad u = \sqrt{3} \; x + s \quad du = \sqrt{3} \; dx$

$$= \frac{\mu_0 I^2}{2\pi} \left[\frac{-\hat{y}}{\sqrt{3}} \left(ln(\sqrt{3}x + s)\right)_0^{a/2} + \hat{x} \ln(y)_{s + \frac{a\sqrt{3}}{2}}^s\right]$$
\[ \vec{F}_{left} = \frac{\mu_o I^2}{2\pi} \left[ -\frac{\hat{y}}{\sqrt{3}} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) + \hat{x} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) \right] \]  

The right wire may be described by the equation (using the same method used for the left wire),

\[ y = -\sqrt{3} x + s + a\sqrt{3} \]

The force on the right wire is,

\[ \vec{F}_{right} = \frac{\mu_o I^2}{2\pi} \left[ \int_{a/2}^{a} -\hat{y} \frac{dx}{\sqrt{3} x + s + a\sqrt{3}} + \int_{s+\frac{a\sqrt{3}}{2}}^{s} \hat{x} \frac{dy}{y} \right] \]

Use substitution \( u = -\sqrt{3} x + s + a\sqrt{3} \) \( du = -\sqrt{3} x \)

\[ \vec{F}_{right} = \frac{\mu_o I^2}{2\pi} \left[ \frac{\hat{y}}{\sqrt{3}} \ln(-\sqrt{3} x + s + a\sqrt{3})_{a/2}^{s} + \hat{x} \ln(y)_{s+\frac{a\sqrt{3}}{2}}^{s} \right] \]  

Sum the forces from equations (16), (19), and (21) to get the total force on the triangular loop.

\[ \vec{F} = \frac{\mu_o I^2}{2\pi} \left[ \frac{a}{s} \frac{\hat{y}}{\sqrt{3}} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) + \hat{x} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) + \frac{\hat{y}}{\sqrt{3}} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) + \hat{x} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) \right] \]

The natural logs sum easily and collecting like components gives,

\[ \vec{F} = \frac{\mu_o I^2}{2\pi} \left[ \hat{x} \ln \left( \frac{s + \frac{a\sqrt{3}}{2}}{s} \right) \cdot \frac{s}{s + \frac{a\sqrt{3}}{2}} + \hat{y} \left( \frac{a}{s} + \frac{1}{\sqrt{3}} \ln \left( \frac{s}{s + \frac{a\sqrt{3}}{2}} \right)^2 \right) \right] \]

We have now proven that the \( x \) terms cancel because \( \ln(1) = 0 \). The final answer for the force on the triangular loop is,

\[ \vec{F} = \frac{\mu_o I^2}{2\pi} \left[ \frac{a}{s} + \frac{2}{\sqrt{3}} \ln \left( \frac{s}{s + \frac{a\sqrt{3}}{2}} \right) \right] \hat{y} \]

Since \( s + \frac{a\sqrt{3}}{2} > s \) the \( \ln \) term is negative. This makes sense because the right-hand rule indicates that the \( y \) component of the forces on the left and right wires points in the \(-\hat{y}\) direction.
Problem 5.13 from Griffiths

Reference figure 5.40 in Griffiths. A long cylindrical wire of radius \( a \) carries a current \( I \). Find the magnetic field inside and outside of the wire for the following conditions:

(a) The current is uniformly distributed over the surface of the wire.
(b) The current is distributed such that \( J \) is proportional to the cylindrical coordinate \( s \), the distance from the axis.

(a) **Uniform \( J \) on surface** The symmetry in this problem makes it a perfect example for using the Guass’ Law of magnetostatics. Technically, it is the integral version of Ampere’s Law. This expression is,

\[
\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} \quad \text{Eq. 5.55 (23)}
\]

To properly use this expression you must choose a loop over which there is a constant magnetic field. In this case we can write the left side of (23) as \( BL \) where \( L \) is the length of the loop. The cylindrical geometry of this problem indicates that the magnetic field must be in \( \hat{\phi} \) direction because that is the only unique direction satisfying the right-hand rule using the current. As such, the Amperian loops we draw in this problem are in the same direction.

Inside the wire our loop encloses no current. Therefore \( I_{\text{enc}} = 0 \) and the magnetic is immediately known to be zero. Moving on to the region outside of the wire we take a loop of radius \( s \) and the current enclosed by it will always be equal to \( I \).

\[
\oint \vec{B} \cdot d\vec{l} = \mu_0 I
\]

\[
B(2\pi s) = \mu_0 I
\]

\[
B = \frac{\mu_0 I}{2\pi s}
\]

This is only the magnitude of the magnetic field. We have already determined the direction based on intuition and knowledge of how it curls around currents. This is exactly the same as the method in which we used Gauss’ Law to solve for the magnitude of the electric field and then just put the direction on the final answer.

\[
\vec{B} = \begin{cases} 
0 & \text{if } s < a \\
\frac{\mu_0 I}{2\pi s} \hat{\phi} & \text{if } s > a
\end{cases}
\]  

(b) **Non-uniform Current Density**

The current density is said to be proportional to the coordinate \( s \). This can be written mathematically as,

\[
J = \alpha s
\]
where \( \alpha \) is an undefined constant. Actually, since we know the dimensions of the wire and the total current in it we can solve for \( \alpha \). Doing this will allow us to solve the rest of the problem.

\[
I = \int \mathbf{J} \cdot d\mathbf{A}
\]

\[
= \int_0^a \int_0^{2\pi} -\alpha s \cdot (sdsd\phi)
\]

\[
= 2\pi\alpha \int_0^a s^2 ds
\]

\[
= \frac{2\pi\alpha a^3}{3}
\]

In these steps it is assumed that \( \mathbf{J} \) is in the \(-\hat{z}\) so that the magnetic field outside the wire will be given in the \( \hat{\phi} \) direction. I then removed the negative sign because we are only solving for the magnitude of \( \alpha \).

\[
\alpha = \frac{3I}{2\pi a^3}
\]

and now we can use the same steps as in Part (a), only this time the \( I_{enc} \) term will be very different.

The left side of (23) is defined from the geometry of the problem and is therefore the same here as in Part (a). Inside the wire,

\[
B(2\pi s) = \mu_0 \int_0^s \frac{3Is}{2\pi a^3} \cdot (2\pi ds)
\]

\[
= \frac{3\mu_0 I}{a^3} \left( \frac{s^3}{3} \right)
\]

\[
= \frac{\mu_0 Is^3}{a^3}
\]

\[
B = \frac{\mu_0 Is^2}{2\pi a^3}
\]

Outside the wire the enclosed is once again the total current \( I \). The magnetic field outside the wire is the same as that found in Part (a) and our final solution is,

\[
\mathbf{B} = \begin{cases} 
\frac{\mu_0 Is^2}{2\pi a^3} \hat{\phi} & \text{s < a} \\
\frac{\mu_0 I}{2\pi s} \hat{\phi} & \text{s > a}
\end{cases}
\]

[5.] Problem 5.15 from Griffiths

Reference figure 5.42 in Griffiths. Two long coaxial solenoids each carry a current \( I \), but in opposite directions. The inner solenoid has a radius \( a \) and \( n_1 \) turns per unit length. The
outer solenoid has radius \( b \) and \( n_2 \) turns per unit length. Find the magnetic field inside the inner solenoid (i), in the region between the two solenoids (ii), and outside both solenoids (iii).

I choose to define the \( z \) axis as pointing to the right in Griffith’s figure. Your solutions may be slightly different if you choose coordinates differently. It is easiest to solve this problem in the opposite order of the way it is written. Examples 5.9 and 5.10 in Griffiths show that the magnetic field outside of a solenoidal or toroidal coil is zero. This is true in general and we can cite this to say,

\[
\vec{B}_{(iii)} = 0
\]

Reference the above figure for the use of Ampere’s law that follows. In the region between the solenoids we only care about the outer solenoid. This is because the magnetic field due to inner solenoid is zero here (this region is the outside of the inner solenoid). Take a square loop (each side of length \( L \)) across some of the windings from the outer solenoid and only the section that resides completely inside the solenoid will contribute to the magnetic field. In terms of (23) we have,

\[
BL = \mu_o I_{enc}
\]

\[
= \mu_o n_2 I L
\]

\[
\vec{B}_{(ii)} = \mu_o n_2 I \hat{z}
\]

where the direction is determined by the right-hand rule and my choice of the \(+\hat{z}\) direction. The magnetic field in the innermost region can be determined using the same method. If our Amperian loop extends from inside the inner solenoid to outside of both then again
only the innermost section contributes to the integral. The current from the inner solenoid is considered negative in this situation because I have already called the outer solenoid’s current positive.

\[ BL = \mu_0(n_2 IL - n_1 IL) \]

\[ = \mu_0 IL(n_2 - n_1) \]

\[ \vec{B}_{(i)} = \mu_0 I(n_2 - n_1) \hat{z} \]  \hspace{1cm} (28)

[6.] Problem 5.16 from Griffiths

Reference figure 5.43 in Griffiths in addition to the figure provided below. A large parallel plate capacitor is moving with a constant speed, \(v\). There is a uniform charge density \(+\sigma\) on the upper plate and \(-\sigma\) on the lower plate.

(a) Find the magnetic field inside and outside the capacitor.

(b) Find the magnetic force density on the upper plate.

(c) Find the speed at which the magnetic force on the plates balances the electric force.

(a) Magnetic field

Figure 4: Illustration of Amperian loops and coordinate system for problem 5.16. The plates are labelled according to surface charge density and their velocity is shown to be coming out of the page. The magnetic fields in each region are labelled with a + or - to indicate which plate from which they result.

The steps here will be entirely analogous to those taken to find the electric fields of a parallel plate capacitor. Example 5.8 in Griffiths provides the method for solving for the field due to each individual plate. Taking a square loop across only the top plate and applying (23) we have,

\[ B_{\text{top}}(2L) = \mu_0 \sigma v L \]

\[ B = \frac{\mu_0 \sigma v}{2} \]
This is the magnitude of $\vec{B}$, the actual field is in opposite directions above and below the plate. Since we are assuming the plates are large the magnetic field does not depend on the distance from them.

The bottom plate is moving in the same direction but has an opposite charge. An opposite charge in the same direction is equivalent to a current in the opposite direction. Therefore, the magnetic field above and below the bottom plate (due to this plate) is in the opposite direction as the upper plate. In the region between the plates this corresponds to similarly directed magnetic fields.

\[ \vec{B}_{\text{inside}} = \mu_0 \sigma v \hat{x} \quad (29) \]

where the $\hat{x}$ direction is perpendicular to $\vec{v}$ and $\vec{v} = v \hat{z}$.

In the region outside the plates the magnetic fields are directed oppositely and cancel.

\[ \vec{B}_{\text{outside}} = 0 \quad (30) \]

(b) Magnetic Force Density on Upper Plate

For a surface current density the magnetic force density is given by,

\[ \vec{f} = \vec{K} \times \vec{B} \quad (31) \]

Surface current density is defined to be the surface charge density multiplied by its velocity.

\[ \vec{K} = \sigma v \hat{z} \quad (32) \]

The force on the upper plate is due only to the magnetic field produced the lower plate. This is another example of how an object cannot exert a force on itself (the electrical analog is that the force on a charge $q$ is given by $q \vec{E}$ where the electric field is the one created by everything except $q$ itself). At the upper plate the magnetic field due to the lower plate is,

\[ \vec{B} = \frac{\mu_0 \sigma v}{2} \hat{x} \]

The force density on the upper plate is,

\[ \vec{f}_{\text{upper}} = \sigma v \hat{z} \times \frac{\mu_0 \sigma v}{2} \hat{x} \]

\[ = \frac{\mu_0 \sigma^2 v^2}{2} \hat{y} \]

(c) Force Balance

The electrical force density on the upper plate is,

\[ \vec{f}_{\text{elec}} = \sigma \vec{E} \]

\[ = -\frac{\sigma^2}{2\epsilon_0} \hat{y} \]

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where the electric field between the plates of a capacitor is known from earlier in the quarter to be \( \vec{E} = -\frac{\sigma}{2\epsilon_0}\hat{y} \). Reference examples 2.4 and 2.5 in Griffiths if you don’t remember how we arrive at this.

Set the magnitudes of these force densities equal to each other and solve for the velocity that is required by this expression.

\[
\vec{f}_{mag} = \vec{f}_{elec}
\]

\[
\frac{\mu_0\sigma^2v^2}{2} = \frac{\sigma^2}{2\epsilon_0}
\]

\[
v^2 = \frac{1}{\mu_0\epsilon_0}
\]

\[
v = \frac{1}{\sqrt{\mu_0\epsilon_0}}
\]

It turns out (as we will see often in relativity) that,

\[
\frac{1}{\sqrt{\mu_0\epsilon_0}} = c
\]

where \( c \) is the speed of light.

The magnetic forces are comparable to the electric forces when the velocity in question reaches the speed of light. This serves as another example that magnetic effects are truly a relativistic phenomena and are generally insignificant compared to electric effects.

[7.] Problem 5.19 from Griffiths

(a) Find the density of free charges in copper. Assume that each copper atom contributes one electron.

(b) Calculate the average electron velocity in a wire of 1mm diameter carrying a current of 1A.

(c) What is the force of attraction between two such wires spaced 1cm apart?

(d) If the wires had no background positive ions then what would be the force of electrical repulsion? How many times stronger than the magnetic force is this?

(a) Free Charge Density

I used the following constants,

\[
\begin{align*}
\text{Density} & = 8920 \frac{\text{kg}}{\text{m}^3} \quad \text{Molar Density} = 63.55 \frac{\text{g}}{\text{mol}} \\
1 \text{ mol} & = 6.02 \times 10^{23} \text{Particiles} \quad 1 \text{ electron} = 1.6 \times 10^{-19} \text{C}
\end{align*}
\]

Find the number of copper atoms per cubic meter,

\[
8920 \frac{\text{kg}}{\text{m}^3} \cdot \frac{1}{63.55} \frac{\text{mol}}{\text{g}} \cdot \frac{1000\text{g}}{\text{kg}} = 1.40 \times 10^5 \frac{\text{mol}}{\text{m}^3}
\]
\[ 1.40 \times 10^5 \text{mol/m}^3 \cdot 6.02 \times 10^{23} \frac{\text{atoms}}{\text{mol}} = 8.45 \times 10^{28} \frac{\text{atoms}}{\text{m}^3} \]

\[ \rho = 8.45 \times 10^{28} \frac{\text{electron}}{\text{m}^3} \cdot 1.60 \times 10^{-19} \frac{\text{C}}{\text{electron}} = 1.35 \times 10^{10} \frac{\text{C}}{\text{m}^3} \]

(b) **Average Electron Velocity**

Currents are measures of how much charge passes through a certain location in a given amount of time. The current in the wire is given to be 1A so we know that 1C of charge passes through a cross section of the wire every second. The cross sectional area of the wire is,

\[ \text{Area} = \pi (0.0005)^2 \text{m}^2 \]
\[ = 7.85 \times 10^{-7} \text{m}^2 \]

The task is to find the length of wire that contains one Coulomb worth of electrons. If we have a direction for the current, then the electron at the end of our length must make it to the cross section within one second. This is how we determine the average velocity.

\[ \frac{1 \text{ electron}}{1.60 \times 10^{-19} \text{C}} = 6.25 \times 10^{18} \frac{\text{electrons}}{\text{C}} \]

Now, knowing how many electrons we must have passing through a cross-section of the wire we can use \( \rho \) to determine how much volume they must occupy.

\[ \text{Volume} = 6.25 \times 10^{18} \frac{\text{electrons}}{\text{C}} \cdot \frac{1 \text{ m}^3}{8.45 \times 10^{28} \text{ electrons}} = 7.40 \times 10^{-11} \text{ m}^3 \]

This is the volume of wire that contains 1C of charge (i.e. 1C worth of electrons). Since the cross section of the wire is set for us we can determine the length of wire that it required to have this many electrons.

\[ \text{Length} = \frac{7.40 \times 10^{-11} \text{ m}^3}{7.85 \times 10^{-7} \text{ m}^2} = 9.43 \times 10^{-5} \text{ m} \]

As stated previously, to maintain a current of 1A the electrons in this length of wire must all pass through a given point in one second. The electrons at the end of the length will then have travelled the entire length in this second. Therefore, the average velocity of the electrons is,

\[ v_{\text{avg}} = 9.43 \times 10^{-5} \frac{\text{m}}{\text{s}} \]

This is indeed a very slow speed. Electronic communication systems are blazingly fast in spite of this. The reason is that communication signals do not depend on specific electrons. When a current is initiated in a wire each electron moves slowly, but they all begin moving at the same time. A current is measured at both ends of the wire at the same time (technically it’s more like the speed of light). It is these currents that constitute signals so we can send messages quickly even though each “messenger” is moving very slowly.
(c) **Magnetic Force Density**

The force density on either of the wires due to the other one is,

\[
\vec{f} = \vec{I} \times \vec{B}
\]  

(33)

The magnetic field in (33) is that due to one of the wires,

\[
B_{\text{wire}} = \frac{\mu_0 I}{2\pi s}
\]

where \(s\) is the distance away from the wire. Here \(s\) is given as 0.01m.

The geometry dictates that \(\vec{I} \perp \vec{B}\) so \(\vec{f}\) is written,

\[
f = IB = (1)\frac{4\pi \times 10^{-7}(1)}{2\pi(0.01)} = 2.0 \times 10^{-5} \text{ N m}
\]

(34)

where you can check the units if you wish.

(d) **Comparison to Electrical Force Density**

The electric force density on a wire of charge density \(\lambda\) due to another line charge is,

\[
\vec{f}_E = \lambda \vec{E}
\]

We know the electric field due to a line charge is,

\[
\vec{E} = \frac{\lambda}{2\pi \epsilon_0 s} \hat{s}
\]

where \(s = 0.01\text{m}\) in this case.

The charge density \(\lambda\) is known from previous work to determine the length of wire that contains 1C of charge.

\[
\lambda = 1.06 \times 10^4 \frac{\text{C}}{\text{m}}
\]

The magnitude of the electrical force density on either wire is,

\[
f = \frac{(1.06 \times 10^4)^2}{2\pi(8.85 \times 10^{-12})(0.01)} = 2.02 \times 19^{20} \frac{\text{N}}{\text{m}}
\]

and again you can check the units to see that this is sensible.

The ratio of electric to magnetic force density is,

\[
\frac{f_E}{f_B} = \frac{2.02 \times 10^{20}}{2 \times 10^{-5}} \approx 10^{25}
\]

All of our previous work with magnetostatics suggested that this number would be large.