

# PHYS 110A - HW #7

Solutions by David Pace

Any referenced equations are from Griffiths

## [1.] Problem 4.11 from Griffiths

A cylinder of length  $L$  and radius  $a$  has a permanent polarization parallel to its axis. Find the bound charge. Sketch the electric field for the following cases: (i)  $L \gg a$ , (ii)  $L \ll a$ , and (iii)  $L \approx a$ .

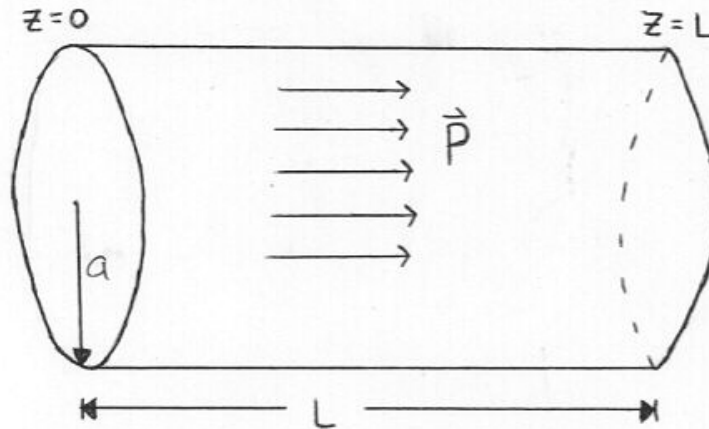


Figure 1: Diagram of Problem 4.11

### Bound Charge

There are two types of bound charge, surface and volume. These are given by, respectively,

$$\sigma_b \equiv \vec{P} \cdot \hat{n} \quad \rho_b \equiv -\vec{\nabla} \cdot \vec{P}$$

There are three surfaces on which to calculate  $\sigma_b$ ; there are two circular ends and the cylindrical surface of length  $L$ . Let one of the circular faces be at position  $z = 0$  and the other at  $z = L$ . The polarization is constant,  $\vec{P} = P\hat{z}$ . Each surface has a different normal vector,  $\hat{n}$ .

$$\text{Circular surface at } z = 0 \quad \rightarrow \quad \sigma_b = P\hat{z} \cdot (-\hat{z}) = -P$$

$$\text{Circular surface at } z = L \quad \rightarrow \quad \sigma_b = P\hat{z} \cdot \hat{z} = P$$

$$\text{Cylindrical surface} \quad \rightarrow \quad \sigma_b = P\hat{z} \cdot \hat{s} = 0$$

The volume bound charge,

$$\rho_b = -\vec{\nabla} \cdot P\hat{z} = 0$$

### Electric Field Sketches

Sketches of the electric field for the various geometries are given below. To make such a drawing consider the polarization vector as a representative of where the positive and negative charge resides. The negative charges are found at the base of this vector and the positive charges are what it points toward. Note that the sketches below represent only the electric fields; inside the cylinder there is a polarization vector that points opposite the arrows I have drawn.

Also, the sketches are represented as cross sections, so the cylinder looks like a rectangle. The radius  $a$  is half of the vertical length of the rectangle.

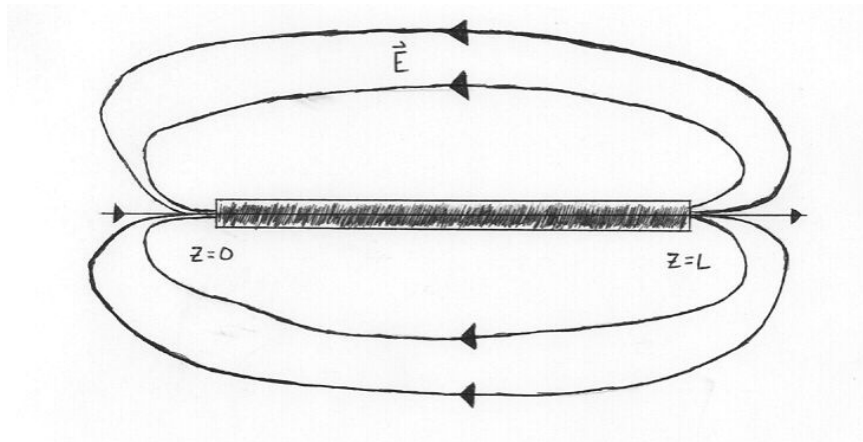


Figure 2: Electric field for  $L \gg a$

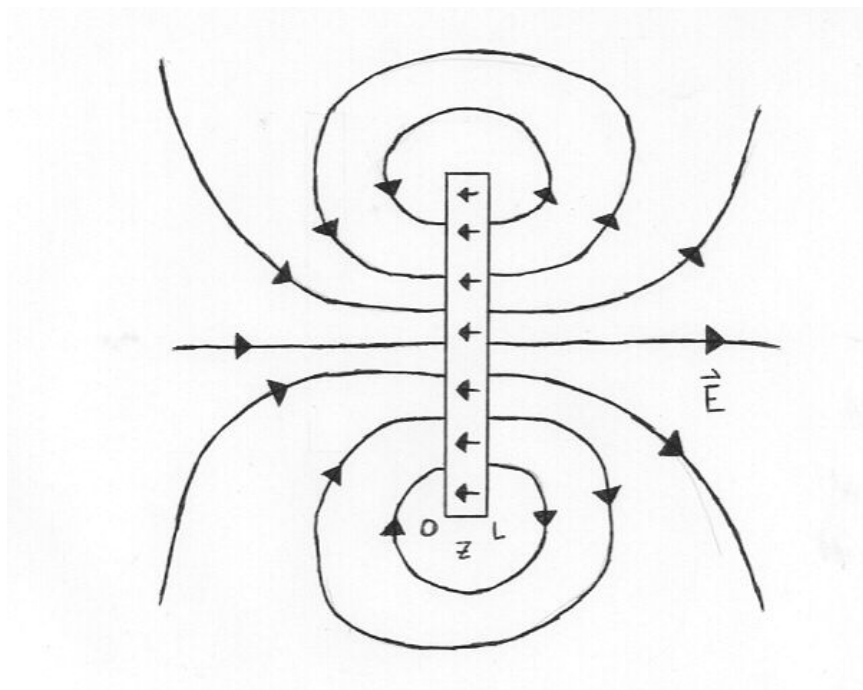


Figure 3: Electric field for  $L \ll a$ . Notice that the electric is uniform inside the cylinder.

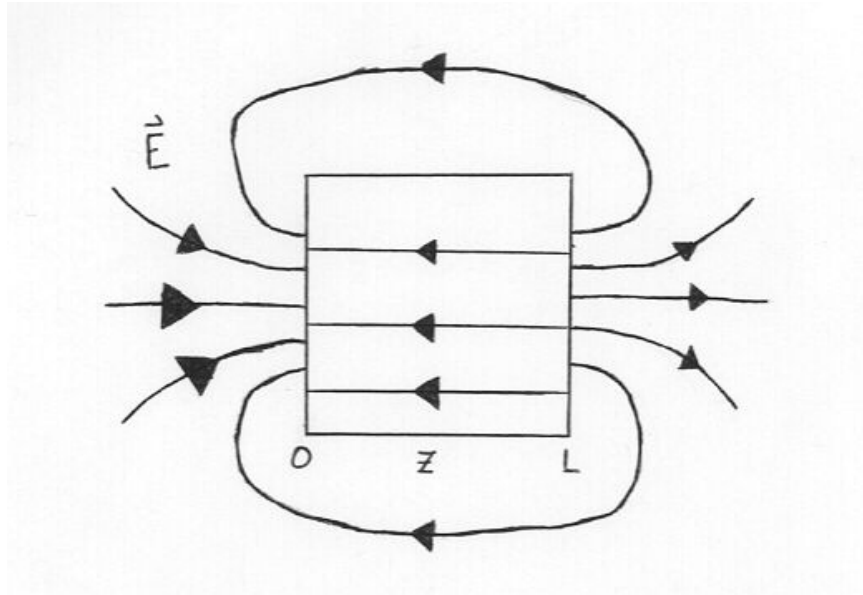


Figure 4: Electric field for  $L \approx a$

**[2.] Problem 4.18 from Griffiths**

Reference Griffiths figure 4.24 for the original diagram. A parallel plate capacitor is filled by two slabs of linear dielectric. Each dielectric has a thickness  $a$ , making the total separation of the capacitor plates  $2a$ . Let slab 1 have a dielectric constant of 2 (i.e.  $\epsilon_{r1} = 2$ ) and slab 2 has a dielectric constant of 1.5 ( $\epsilon_{r2} = 1.5$ ). The free charge densities on the capacitor plates (as drawn in Griffiths) are  $\sigma_f = +\sigma$  on the plate near slab 1 and  $\sigma_f = -\sigma$  on the plate nearer slab 2.

- (a) Find the electric displacement in each slab.
- (b) Find the electric field in each slab.
- (c) Find the polarization in each slab.
- (d) Find the potential difference between the plates.
- (e) Determine the location and amount of any bound charge.
- (f) Having described all of the free and bound charge recalculate the electric field in each slab. Make sure it agrees with the answer you found earlier.

Figure 5 represents the capacitor in this problem. It is drawn vertically so that I can reference left and right hand sides throughout the solution. This should make applications of Gauss' Law easier to follow. You also see the orientation of the z-axis and the directions on the normal vectors from the slabs.

**(a) Electric Displacement**

The electric displacement may be found using the appropriate form of Gauss' Law.

$$\oint \vec{D} \cdot d\vec{a} = Q_{f,enc} \quad \text{Eq. 4.23}$$

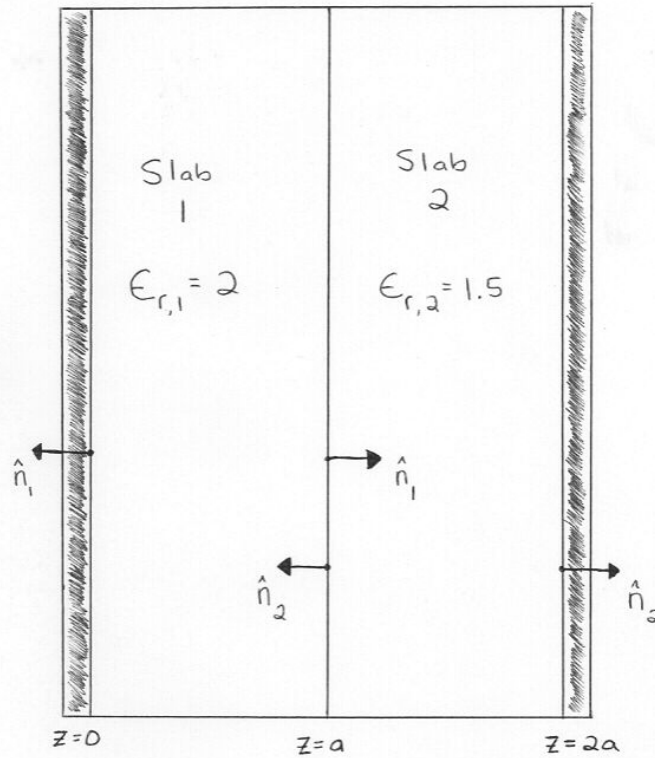


Figure 5: Diagram of Problem 4.18. Here it is drawn vertically to distinguish left and right sides.

where  $Q_{f,enc}$  is the free charge enclosed within the surface of the integral.

The procedure is the same as that used earlier in the quarter for finding the electric field inside a parallel plate capacitor (if you want to review, see Griffiths, example 2.4 on page 73). Beginning with the plate of  $\sigma_f = +\sigma$  we can draw a cylindrical Gaussian surface across the boundary and see that,

$$\vec{D} = \frac{\sigma}{2} \hat{z}$$

This is the contribution to  $\vec{D}$  only from the  $\sigma_b = +\sigma$  plate. The  $\sigma_b = -\sigma$  makes a contribution of similar amplitude in the same direction, therefore,

$$\vec{D}_1 = \vec{D}_2 = \sigma \hat{z}$$

### (b) Electric Field

Once we have solved for the electric displacement in an object it is possible to find the electric field using,

$$\vec{D} = \epsilon \vec{E} \quad \text{Eq. 4.32}$$

where  $\epsilon \equiv \epsilon_o(1 + \chi_e)$  (Eq. 4.33). The problem gives us the dielectric constants of each slab; this value is  $\epsilon_r = 1 + \chi_e$ . While the electric displacement was constant throughout the interior of the capacitor the electric field depends on the dielectric materials present. Recall, the dielectric will polarize according to the strength of the electric field and this will in turn affect  $\vec{E}$ .

$$\vec{E}_1 = \frac{\vec{D}_1}{\epsilon_{r1}\epsilon_o} = \frac{\sigma}{2\epsilon_o}$$

$$\vec{E}_2 = \frac{\vec{D}_2}{\epsilon_{r2}\epsilon_o} = \frac{\sigma}{1.5\epsilon_o} = \frac{2\sigma}{3\epsilon_o}$$

**(c) Polarization**

$$\vec{P} = \epsilon_o\chi_e\vec{E} \quad \text{Eq. 4.30}$$

and from part (b) we know  $\chi_e = \epsilon_r - 1$ .

$$\vec{P}_1 = \epsilon_o(2 - 1)\frac{\sigma}{2\epsilon_o}\hat{z} = \frac{\sigma}{2}\hat{z}$$

$$\vec{P}_2 = \epsilon_o(1.5 - 1)\frac{2\sigma}{3\epsilon_o}\hat{z} = \frac{\sigma}{3}\hat{z}$$

**(d) Potential Between the Plates**

The electric field we have found is the total field, i.e. it accounts for the dielectrics and therefore the potential between the plates is given by the usual line integral expression,

$$V = - \int \vec{E} \cdot d\vec{l}$$

$$\begin{aligned} V_{plates} &= - \int_a^0 E_1 dz - \int_{2a}^a E_2 dz \\ &= - \frac{\sigma}{2\epsilon_o} z \Big|_a^0 - \frac{2\sigma}{3\epsilon_o} z \Big|_{2a}^a = - \frac{\sigma}{2\epsilon_o} (0 - a) - \frac{2\sigma}{3\epsilon_o} (a - 2a) \\ &= \frac{a\sigma}{2\epsilon_o} + \frac{2a\sigma}{3\epsilon_o} = \frac{7a\sigma}{6\epsilon_o} \end{aligned}$$

**(e) Bound Charge**

Volume bound charge,

$$\rho_{b1} = -\vec{\nabla} \cdot \frac{\sigma}{2}\hat{z} = 0$$

$$\rho_{b2} = -\vec{\nabla} \cdot \frac{\sigma}{3}\hat{z} = 0$$

The surface charge must be calculated at all surfaces the electric fields pass through, this is why we don't worry about the outside edges of the slabs.

$$\text{at } z = 0 \quad \sigma_{b1} = \frac{\sigma}{2}\hat{z} \cdot -\hat{z} = -\frac{\sigma}{2}$$

$$\text{at } z = a \quad \sigma_{b1} = \frac{\sigma}{2}\hat{z} \cdot \hat{z} = \frac{\sigma}{2}$$

$$\text{also at } z = a \quad \sigma_{b2} = \frac{\sigma}{3}\hat{z} \cdot -\hat{z} = -\frac{\sigma}{3}$$

$$\text{at } z = 2a \quad \sigma_{b2} = \frac{\sigma}{3}\hat{z} \cdot \hat{z} = \frac{\sigma}{3}$$

### (f) Determine Electric Fields from Charge

If we consider all of the surface charge, bound and free, then the electric field between the plates is given by,

$$\vec{E} = \frac{\sigma}{\epsilon_0}$$

where  $\sigma$  is the total charge density of the surfaces and there is no dependence on the dielectrics because we have accounted for them in computing the bound charge densities.

Find the total charge density on each side of the dielectric. For slab 1,

$$\sigma_{left} = \sigma - \frac{\sigma}{2} = \frac{\sigma}{2}$$

$$\sigma_{right} = \frac{\sigma}{2} - \frac{\sigma}{3} + \frac{\sigma}{3} - \sigma = -\frac{\sigma}{2}$$

Slab 1 has a charge density  $\frac{\sigma}{2}$  and the field is then,

$$\vec{E}_1 = \frac{\sigma}{2\epsilon_0} \hat{z}$$

where the direction is based on the orientation of the plates and the location of the positive charge.

For slab 2,

$$\sigma_{left} = \sigma - \frac{\sigma}{2} + \frac{\sigma}{2} - \frac{\sigma}{3} = \frac{2\sigma}{3}$$

$$\sigma_{right} = \frac{\sigma}{3} - \sigma = -\frac{2\sigma}{3}$$

and then this electric field follows,

$$\vec{E}_2 = \frac{2\sigma}{3\epsilon_0} \hat{z}$$

Both solutions match the ones we found earlier.

### [3.] Problem 4.24 from Griffiths

A neutral conducting sphere of radius  $a$  is surrounded by a dielectric shell with dielectric constant  $\epsilon_r$ . The total radius of the dielectric is  $b$ . This entire object is then placed in a uniform background electric field of value  $\vec{E}_0$ . Find the electric field inside the dielectric shell.

We are correct in assuming that finding the electric field right away is difficult. Techniques used previously to find the potential at all points in space still apply here, however, and then the electric field can be computed from that. This is a boundary value problem so we start by writing out the boundary conditions. Since the sphere is a conductor we know the potential is constant throughout it. There is a background electric field so the potential

cannot be set to zero at infinity. Setting the potential of the conducting shell to zero is the first step in our solution.

$$V_{di} = \text{Potential within dielectric} \quad V_{out} = \text{Potential outside of dielectric}$$

### Boundary Conditions

$$(i) \quad V(r \leq a, \theta) = 0 \quad \rightarrow \quad V_{di}(a, \theta) = 0 \quad \text{Potentials must be equal at boundaries}$$

$$(ii) \quad V_{di}(b, \theta) = V_{out}(b, \theta) \quad \text{Same reasoning as above}$$

$$(iii) \quad \epsilon_{out} \frac{\partial V_{out}}{\partial n} - \epsilon_{in} \frac{\partial V_{in}}{\partial n} = -\sigma_f \quad \text{Eq. 4.41}$$

$$\epsilon_{out} \frac{\partial V_{out}}{\partial r} - \epsilon_{di} \frac{\partial V_{di}}{\partial r} = 0 \quad \text{For our problem}$$

$$\epsilon_o \frac{\partial V_{out}}{\partial r} = \epsilon_r \epsilon_o \frac{\partial V_{di}}{\partial r}$$

$$\frac{\partial V_{out}}{\partial r} = \epsilon_r \frac{\partial V_{di}}{\partial r} \quad \text{At } r = b$$

$$(iv) \quad V_{out}(r \rightarrow \infty, \theta) = -E_o r \cos \theta$$

Since there is no free charge in the dielectric we may treat the potential as a solution to Laplace's equation just as we did with conductors previously (review Griffiths p. 186 for the explanation). Now this problem is a separation of variables and the general solutions for the potentials are,

$$V_{di}(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$V_{out}(r, \theta) = -E_o r \cos \theta + \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos \theta)$$

For the dielectric we must keep both terms because it does not contain the  $r = 0$  or the  $r = \infty$  positions that usually set one of them to zero. The outside potential is given in terms of the constant  $C_l$  because these are not necessarily the same as the  $B_l$  terms found in the dielectric expression. We must begin with generic variables and let the mathematics sort out their values. The  $E_o r \cos \theta$  term represents the background field (accounting for condition (iv)).

From (i),

$$V_{di}(a, \theta) = 0 = \sum_{l=0}^{\infty} \left( A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos \theta)$$

None of the Legendre polynomials are zero so for each value of  $l$  the factor in parentheses must be zero.

$$A_l a^l + \frac{B_l}{a^{l+1}} = 0$$

$$B_l = -A_l a^{2l+1}$$

From (ii),

$$\sum_{l=0}^{\infty} \left( A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos \theta) = -E_o b \cos \theta + \sum_{l=0}^{\infty} \frac{C_l}{b^{l+1}} P_l(\cos \theta)$$

Equate like terms of the Legendre polynomials ( $P_1(\cos \theta) = \cos \theta$ ).

$$A_l b^l + \frac{B_l}{b^{l+1}} = \frac{C_l}{b^{l+1}} \quad \text{for } l \neq 1$$

$$A_1 b + \frac{B_1}{b^2} = -E_o b + \frac{C_1}{b^2} \quad \text{for } l = 1$$

The  $l \neq 1$  case, use the  $B_1$  substitution found above,

$$A_l b^l + \frac{-A_l a^{2l+1}}{b^{l+1}} = \frac{C_l}{b^{l+1}}$$

$$C_l = A_l \left[ b^{2l+1} - a^{2l+1} \right]$$

The  $l = 1$  case,

$$A_1 b - \frac{A_1 a^3}{b^2} = -E_o b + \frac{C_1}{b^2}$$

$$C_1 = A_1 (b^3 - a^3) + E_o b^3$$

From (iii),

$$\begin{aligned} \epsilon_r \frac{\partial V_{di}}{\partial r} \Big|_{r=b} &= \epsilon_r \sum_{l=0}^{\infty} \left( l A_l r^{l-1} + \frac{-(l+1)B_l}{r^{l+2}} \right) P_l(\cos \theta) \Big|_{r=b} \\ &= \epsilon_r \sum_{l=0}^{\infty} \left( l A_l b^{l-1} + \frac{-(l+1)B_l}{b^{l+2}} \right) P_l(\cos \theta) \end{aligned}$$

$$\frac{\partial V_{out}}{\partial r} \Big|_{r=b} = -E_o P_1(\cos \theta) + \sum_{l=0}^{\infty} \left( \frac{-(l+1)C_l}{b^{l+2}} \right) P_l(\cos \theta)$$



These quantities are equal. Comparing like terms for  $l = 1$  gives,

$$\begin{aligned}\epsilon_r \left[ A_1 - \frac{2B_1}{b^3} \right] &= -E_o - \frac{2C_1}{b^3} \\ \epsilon_r \left[ A_1 - \frac{2}{b^3} (-A_1 a^3) \right] &= -E_o - \frac{2}{b^3} [A_1(b^3 - a^3) + E_o b^3]\end{aligned}$$

...more algebra...

$$A_1 = \frac{-3E_o}{\epsilon_r \left( 1 + \frac{2a^3}{b^3} \right) + \frac{2(b^3 - a^3)}{b^3}}$$

For the  $l \neq 1$  terms,

$$\begin{aligned}lA_l b^{l-1} - \frac{(l+1)B_l}{b^{l+2}} &= -\frac{(l+1)C_l}{b^{l+2}} \\ lA_l b^{l-1} - \frac{(l+1)}{b^{l+2}} (-A_l a^{2l+1}) &= \frac{-(l+1)}{b^{l+2}} A_l [b^{2l+1} - a^{2l+1}]\end{aligned}$$

Factor out  $A_l$  from both sides. It seems like we are getting nowhere because now our expression has no way of solving for  $A_l$ . It turns out that we will still get valuable information from the outcome.

$$\begin{aligned}lb^{l-1} + \frac{(l+1)a^{2l+1}}{b^{l+2}} &= \frac{-(l+1)}{b^{l+2}} [b^{2l+1} - a^{2l+1}] \\ &= \frac{-(l+1)b^{2l+1}}{b^{l+2}} + \frac{(l+1)a^{2l+1}}{b^{l+2}}\end{aligned}$$

$$lb^{l-1} = \frac{-(l+1)b^{2l+1}}{b^{l+2}}$$

$$lb^{2l+1} = -(l+1)b^{2l+1}$$

$$l = -(l+1)$$

$$l = -\frac{1}{2}$$

This means that for  $l = -\frac{1}{2}$  the  $A_{1/2}$  can have any value. Our general solution for the potential requires that  $l$  has a range of values, all of which are whole, positive numbers. The only time this equation is satisfied occurs for the trivial solution  $A_l = 0$ . The other constants are proportional to  $A_l$  so we may say,

$$\text{For } l \neq 1 \quad A_l = B_l = C_l = 0$$

The potential within the dielectric has only the  $l = 1$  term.

$$\begin{aligned} V_{di}(r, \theta) &= \left( A_1 r + \frac{B_1}{r^2} \right) P_1(\cos \theta) \\ &= A_1 \left( r - \frac{a^3}{r^2} \right) P_1(\cos \theta) \quad \text{substituting for } B_1 \end{aligned}$$

The electric field is the negative gradient of the potential. In spherical coordinates this gives us a final answer,

$$\begin{aligned} \vec{E}_{di} &= -A_1 \vec{\nabla} \left[ r - \frac{a^3}{r^2} \right] P_1(\cos \theta) \\ &= -A_1 \left[ \hat{r} P_1(\cos \theta) \left( 1 + \frac{2a^3}{r^3} \right) + \hat{\theta} \frac{1}{r} \left( r - \frac{a^3}{r^2} \right) (-\sin \theta) \right] \end{aligned}$$

where  $A_1$  has been written out above.

#### [4.] Problem 4.26 from Griffiths

A spherical conductor has radius  $a$  and a charge  $Q$ . It is surrounded by a dielectric material of susceptibility  $\chi_e$  and total radius  $b$ . Use Eq. 4.58 to find the energy of this configuration.

$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau \quad \text{Eq. 4.58}$$

Find  $\vec{D}$  first since it only depends on the free charge and we know that since the sphere is a conductor the free charge  $Q$  must reside on its surface.

$$\vec{D} = \begin{cases} 0 & \text{for } 0 \leq r \leq a \\ \frac{Q}{4\pi r^2} \hat{r} & \text{for } a \leq r \leq \infty \end{cases}$$

The electric field is found from,

$$\vec{E} = \frac{\vec{D}}{\epsilon_o(1 + \chi_e)}$$

and this depends on the presence of dielectrics due to the  $\chi_e$  term.

$$\vec{E} = \begin{cases} 0 & \text{for } 0 \leq r \leq a \\ \frac{Q}{4\pi\epsilon_o(1+\chi_e)r^2} \hat{r} & \text{for } a \leq r \leq b \\ \frac{Q}{4\pi\epsilon_o r^2} \hat{r} & \text{for } b \leq r \leq \infty \end{cases}$$

Now the energy may be found from the integration for each region of differing  $\vec{D}$  and  $\vec{E}$  values. These integrals are over all space, but since the integrands have no  $\theta$  or  $\phi$  dependence I have factored out a  $4\pi$  and written the remaining integrals over  $r$  only. Remember

that  $d\tau = r^2 \sin \theta dr d\theta d\phi$ .

$$\begin{aligned}
 W &= \frac{1}{2}(4\pi) \left[ \int_0^a 0 \cdot 0 r^2 dr + \int_a^b \frac{Q\hat{r}}{4\pi r^2} \cdot \frac{Q\hat{r}}{4\pi\epsilon_o(1+\chi_e)r^2} r^2 dr + \int_b^\infty \frac{Q\hat{r}}{4\pi r^2} \cdot \frac{Q\hat{r}}{4\pi\epsilon_o r^2} r^2 dr \right] \\
 &= 2\pi \frac{Q^2}{16\pi\epsilon_o} \left[ \int_a^b \frac{dr}{(1+\chi_e)r^2} + \int_b^\infty \frac{dr}{r^2} \right] \\
 &= \frac{Q^2}{8\pi\epsilon_o} \left[ \frac{1}{1+\chi_e} \left( -\frac{1}{r} \Big|_a^b + \left( -\frac{1}{r} \Big|_b^\infty \right) \right] \\
 &= \frac{Q^2}{8\pi\epsilon_o} \left[ \frac{1}{(1+\chi_e)a} - \frac{1}{(1+\chi_e)b} + \frac{1}{b} \right] \\
 &= \frac{Q^2}{8\pi\epsilon_o(1+\chi_e)} \left[ \frac{1}{a} + \frac{\chi_e}{b} \right]
 \end{aligned}$$

### [5.] Problem 4.28 from Griffiths

Reference figure 4.32 in Griffiths. A cylindrical capacitor (inner radius  $a$  and outer radius  $b$ ) is placed in a tank of dielectric oil. The oil has a susceptibility of  $\chi_e$  and a mass density  $\rho$ . If the potential of the inner cylinder is held constant at  $V$  and the outer cylinder is grounded, then to what height,  $h$ , will the oil rise within in the cylinders?

This problem will be solved by finding the force of the capacitor fields on the oil and comparing them to the gravitational force on the liquid. These forces will be equal when the oil reaches its equilibrium height in the capacitor. The force on the oil due to the capacitor fields is,

$$F = -\frac{dW}{dx} = -\frac{V^2}{2} \frac{dC}{dx}$$

The sign of the force does not matter for this problem because we are going to equate the forces to find the equilibrium position. Also, it is  $h$  that is our variable instead of the generic variable  $x$ . The force on the oil due to the fields is,

$$F = \frac{V^2}{2} \frac{dC}{dh}$$

The magnitude of the force on the oil due to gravity is the mass of the oil multiplied by the acceleration due to gravity ( $g$ ).

$$F_{grav} = mg = \rho\pi(b^2 - a^2)hg$$

The next step is to determine the capacitance of the system in terms of  $h$ . Earlier in the quarter we found the capacitance of an ideal cylindrical capacitor to be,

$$C_{cyl} = \frac{2\pi\epsilon_o l}{\ln \frac{b}{a}}$$

where  $l$  is the total length of the cylinders.

Consider this to be a system of two capacitors (not as a circuit, but simply deal with the oil region and the air region separately). The potential within the capacitor region where there is no oil is given by,

$$\begin{aligned} V_{air} &= \frac{Q_{air}}{C_{air}} = \frac{Q_{air} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon_o l} \\ &= \frac{\sigma_{air} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon_o} \end{aligned}$$

where  $\sigma_{air}$  is the charge density on the cylinders.

For the oil region, and by similar argument,

$$\begin{aligned} V_{oil} &= \frac{Q_{oil}}{C_{oil}} = \frac{Q_{oil} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon_o l} \\ &= \frac{\sigma_{oil} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon} \end{aligned}$$

Our goal at this point is to write the capacitance of this system in terms of  $h$ . Then we will be able to treat this as a force balance problem and compare the force of gravity on the oil with the force of the capacitor on the oil. I have written the charge densities as  $\sigma$  because the actual charge they represent is spread across the surface of their respective cylinder. The values are still in units of charge of per unit length, as is common procedure for dealing with cylindrical capacitors. Knowing the potential between the cylinders is constant we want to find the total capacitance from,

$$C = \frac{Q_{tot}}{V}$$

This requires finding  $Q_{tot}$ . We have already described the charge densities of this problem and may now use them to find  $Q_{tot}$ .

$$Q_{tot} = \sigma_{oil}h + \sigma_{air}(l - h)$$

We need to relate the charge densities in order to put  $Q_{tot}$  in a useable form.

The problem states that these potentials are equal, they are both  $V$ . This allows us to relate the charge densities,

$$\begin{aligned} \frac{\sigma_{air} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon_o} &= \frac{\sigma_{oil} \ln\left(\frac{b}{a}\right)}{2\pi\epsilon} \\ \frac{\sigma_{air}}{\epsilon_o} &= \frac{\sigma_{oil}}{\epsilon} \\ \sigma_{oil} &= \epsilon_r \sigma_{air} \end{aligned}$$

Now rewrite  $Q_{tot}$ ,

$$\begin{aligned} Q_{tot} &= h(\epsilon_r \sigma_{air} - \sigma_{air}) + l \sigma_{air} \\ &= \sigma_{air}(h(\epsilon_r - 1) + l) \\ &= \sigma_{air}(h\chi_e + l) \end{aligned}$$

Since the potential is the same in each region we can use either value for calculating  $C$ . I wrote the total charge in terms  $\sigma_{air}$  above so I will use that value here.

$$\begin{aligned} C &= \frac{Q_{tot}}{V} = \frac{\sigma_{air}(h\chi_e + l)}{\frac{\sigma_{air} \ln \frac{b}{a}}{2\pi\epsilon_o}} \\ &= \frac{2\pi\epsilon_o(h\chi_e + l)}{\ln \frac{b}{a}} \end{aligned}$$

The magnitude of the force on the oil due to the capacitor fields is given by,

$$\begin{aligned} F &= \frac{V^2}{2} \frac{dC}{dh} = \frac{V^2}{2} \frac{d}{dh} \left[ \frac{2\pi\epsilon_o(h\chi_e + l)}{\ln \frac{b}{a}} \right] \\ &= \frac{V^2 \pi \epsilon_o \chi_e}{\ln \frac{b}{a}} \end{aligned}$$

Equate the forces and solve for  $h$ ,

$$\begin{aligned} F_{grav} &= F \\ \rho\pi(b^2 - a^2)hg &= \frac{V^2 \pi \epsilon_o \chi_e}{\ln \frac{b}{a}} \end{aligned}$$

$$h = \frac{\epsilon_o \chi_e V^2}{\rho(b^2 - a^2)g \ln \frac{b}{a}}$$

#### [6.] Problem 4.32 from Griffiths

A point charge  $q$  is placed at the center of a dielectric sphere. The sphere has susceptibility  $\chi_e$  and radius  $R$ . Find the electric field, polarization, and bound charge densities. What is the total bound charge on the surface of the sphere? Where is the compensating negative bound charge?

##### (a) Electric Field

Find  $\vec{D}$  first,

$$\vec{D} = \frac{q}{4\pi r^2} \hat{r} = \epsilon_o(1 + \chi_e) \vec{E}$$

Again the electric field is different inside the dielectric than outside.

$$\vec{E} = \begin{cases} \frac{q}{4\pi\epsilon_o(1+\chi_e)r^2}\hat{r} & 0 < r \leq R \\ \frac{q}{4\pi\epsilon_or^2}\hat{r} & R \leq r \leq \infty \end{cases}$$

**(b) Polarization**

$$\vec{P} = \epsilon_o\chi_e\vec{E}$$

$$\vec{P} = \begin{cases} \frac{\chi_eq}{4\pi(1+\chi_e)r^2}\hat{r} & 0 < r \leq R \\ 0 & R \leq r \end{cases}$$

**(c) Bound Charge**

At the surface  $R = r$ ,

$$\begin{aligned} \sigma_b = \vec{P} \cdot \hat{n} &= \frac{\chi_eq}{4\pi(1+\chi_e)r^2}\hat{r} \cdot \hat{r} \\ &= \frac{\chi_eq}{4\pi(1+\chi_e)R^2} \end{aligned}$$

To determine what happened to the negative bound surface (it must exist to cancel the surface charge we just computed in order that the dielectric remain charge neutral) we will look at the bound volume charge.

$$\begin{aligned} \rho_b = -\vec{\nabla} \cdot \vec{P} &= -\vec{\nabla} \cdot \frac{\chi_eq}{4\pi(1+\chi_e)r^2}\hat{r} \\ &= -\frac{\chi_eq}{4\pi(1+\chi_e)}\vec{\nabla} \cdot \frac{\hat{r}}{r^2} \\ &= -\frac{\chi_eq}{1+\chi_e}\delta^3(\vec{r}) \end{aligned}$$

This made use of,

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi\delta^3(\vec{r}) \quad \text{Eq. 1.99}$$

The delta function in the expression for the volume bound charge indicated that there must exist an additional bound surface charge at the position  $\vec{r} = 0$ . We can explain this as a consequence of the point-like nature of the charge  $q$  at the center. It attracts a  $-q$  charge buildup near it, but since it has no dimensions (the definition of “point-like”) there is no real surface to spread this bound charge along.

**[7.] Problem 4.33 from Griffiths**

Reference figure 4.34 in Griffiths. Electric fields bend as they cross interfaces of different dielectrics. Assuming there is no free charge at the boundary show that,

$$\frac{\tan \theta_2}{\tan \theta_1} = \frac{\epsilon_2}{\epsilon_1} \quad \text{Eq. 4.68}$$

In addition, consider a dielectric lens. Would a convex lens made of dielectric material tend to focus or defocus the electric field?

Electric fields behave according to well established boundary conditions across material interfaces. The parallel components of  $\vec{E}$  are given simply,

$$E_{\parallel,1} = E_{\parallel,2} \quad \text{Eq. 4.29, originally Eq. 2.32}$$

The perpendicular components follow a slightly more complicated relation,

$$\epsilon_1 E_{\perp,1} - \epsilon_2 E_{\perp,2} = \sigma_f$$

This problem sets  $\sigma_f = 0$  so the previous expression simplifies,

$$\epsilon_1 E_{\perp,1} = \epsilon_2 E_{\perp,2}$$

Next we rewrite the electric fields of the problem as parallel and perpendicular components.

$$\begin{aligned} E_{\parallel,1} &= E_1 \sin \theta_1 & E_{\parallel,2} &= E_2 \sin \theta_2 \\ E_{\perp,1} &= E_1 \cos \theta_1 & E_{\perp,2} &= E_2 \cos \theta_2 \end{aligned}$$

The tangents of each angle are then given by,

$$\begin{aligned} \frac{E_{\parallel,1}}{E_{\perp,1}} &= \frac{E_1 \sin \theta_1}{E_1 \cos \theta_1} = \tan \theta_1 \\ \frac{E_{\parallel,2}}{E_{\perp,2}} &= \frac{E_2 \sin \theta_2}{E_2 \cos \theta_2} = \tan \theta_2 \end{aligned}$$

Now solve for the ratio of  $\tan \theta$  terms.

$$\begin{aligned} \frac{\tan \theta_2}{\tan \theta_1} &= \frac{E_{\parallel,2}}{E_{\perp,2}} \cdot \frac{E_{\perp,1}}{E_{\parallel,1}} \\ &= \frac{E_{\perp,1}}{E_{\perp,2}} \quad \text{since the parallel components are equal} \\ &= \frac{\frac{\epsilon_2}{\epsilon_1} E_{\perp,2}}{E_{\perp,2}} \\ &= \frac{\epsilon_2}{\epsilon_1} \end{aligned}$$

Refer back to figure 4.34 for the lens question. If this setup represents a lens we can say that region 1 is air and region 2 is the lens. Now we know that  $\epsilon_1 = \epsilon_o$  and that  $\epsilon_2 > \epsilon_o$ . The ratio of tangents will be greater than 1, meaning that the electric field in region 2 has a proportionally larger parallel component. This larger parallel component means that the electric field is being bent away from the normal (the normal is the line we use to measure the angle  $\theta$ ). Since the electric field is being bent away from the normal this lens is defocusing the field lines.