[1.] Problem 6.8 from Griffiths

A long cylinder has radius $R$ and a magnetization given by $\vec{M} = ks^2 \hat{\phi}$. Figure 6.13 in Griffiths represents this cylinder and $k$ is a constant. For points inside and outside the cylinder find the magnetic field due to $\vec{M}$.

Solution

The magnetic field due to $\vec{M}$ is that resulting from the bound currents that $\vec{M}$ generates. These currents are related to the magnetization by,

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \quad \text{Eq. 6.13}$$

$$\vec{K}_b = \vec{M} \times \hat{n} \quad \text{Eq. 6.14} \quad (1)$$

It should be immediately noted that $\vec{J}_b = 0$ outside of the cylinder (i.e. for $s > R$) because the magnetization is zero in that region. The bound volume current inside the cylinder may be found using the expression for curl in cylindrical coordinates. All of the terms except one are zero.

$$\vec{J}_b = \vec{\nabla} \times ks^2 \hat{\phi} \quad (2)$$

$$= \frac{1}{s} \left[ \frac{\partial}{\partial s} (s \cdot ks^2) \right] \hat{z} \quad (3)$$

$$= \frac{1}{s} (3ks^2) \hat{z} \quad (4)$$

$$= 3ks \hat{z} \quad (5)$$

To find the bound surface current use $\hat{n} = \hat{s}$. The problem describes this as a long cylinder, which means that end effects may be neglected. This surface current is only defined at the surface where $s = R$.

$$\vec{K}_b = kR^2 \hat{\phi} \times \hat{s} \quad (6)$$

$$= -kR^2 \hat{z} \quad (7)$$

since $\hat{\phi} \times \hat{s} = -\hat{z}$.

All of the current is directed along the axis of the cylinder so this problem exhibits symmetry. Ampere’s law may be used to find the magnetic field.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad (8)$$

where $I_{enc}$ includes both free and bound currents. The left side of (8) will always be equal to $B_\phi(2\pi s)$ because the magnetic field must always be in the $\hat{\phi}$ direction. This problem
reduces to a matter of finding the magnetic field a certain distance, \( s \), away from a current-carrying wire. 

For the region inside the cylinder, \( s < R \), the enclosed current is only that due to the bound volume current density.

\[
I_{\text{enc}} = \int_{0}^{s} \vec{J}_b \cdot d\vec{a} \\
= \int_{0}^{s} 3ks\hat{z} \cdot s \, ds \, d\phi \\
= 3k(2\pi)\frac{s^3}{3} = 2\pi ks^3
\]

Using (8) and (11) the magnetic field inside the cylinder is,

\[
\vec{B}_{s<R} = \mu_0ks^2 \hat{\phi}
\]

The total enclosed current for the region outside the cylinder, \( s > R \), considers the bound volume and bound surface current contributions. The current on the surface is given by the surface current density times the circumference of the cylinder, \( l \).

\[
I_{\text{enc}} = \int_{0}^{R} \vec{J}_b \cdot d\vec{a} + K_b l \\
= \int_{0}^{R} 3ks \cdot s \, ds \, d\phi + (-kR^2)(2\pi R) \\
= 3k(2\pi)\frac{R^3}{3} - 2\pi kR^3 \\
= 0
\]

Since the total enclosed current is zero, the magnetic field is also zero.

\[
\vec{B}_{s>R} = 0
\]

[2.] Problem 6.9 from Griffiths

A short cylinder of length, \( L \), and radius, \( a \), features a permanent magnetization of \( \vec{M} \). This magnetization is parallel to the cylinder axis. Find the bound currents and sketch the magnetic field of this cylinder for the geometries where \( L \ll a \), \( L \gg a \), and \( L \approx a \).

**Solution**

Being uniform and parallel to the cylinder’s axis means that the magnetization may be written as,

\[
\vec{M} = M_o\hat{z}
\]

where \( M_o \) is a constant. The bound currents may be found using (1),

\[
\vec{J}_b = \vec{\nabla} \times M_o\hat{z} = 0
\]

\[
\vec{K}_b = M_o\hat{z} \times \hat{s} = M_o\hat{\phi}
\]
where \( \hat{n} = \hat{s} \) for the surface current density because the relevant surface is the side of the cylinder.

**Figure 1: \( L \gg a \):** Although not clearly indicated by the drawing, the magnetic field inside the cylinder is aligned with the magnetization that is shown. This field is uniform inside the cylinder.

**Figure 2: \( L \ll a \):** In this case the cylinder looks like a single loop of current. The field in this configuration is essentially that of a physical magnetic dipole.
[3.] Professor Carter Problem

An infinitely long cylinder of radius \(a\) has its axis along the \(z\)-axis. Its magnetization is given in cylindrical coordinates by \(\vec{M} = M_o (s/a)^2 \hat{\phi}\), where \(M_o\) is a constant. Find the bound currents and verify that the net charge transferred by the current is zero. Find \(\vec{B}\) everywhere in space using two methods: (a) Calculate \(\vec{B}\) using the calculated bound currents, and (b) Use the auxiliary field, \(\vec{H}\), and the modified form of Ampere’s law to find \(\vec{B}\).

Solution

Use (1) to solve for the bound currents. Since this is an infinitely long cylinder, the only surface is the one at \(s = a\).

\[
\vec{K}_{b,s=a} = \frac{M_o a^2}{a^2} \hat{\phi} \times \hat{s} \quad (21)
\]

\[
= -M_o \hat{z} \quad (22)
\]

The magnetization has only a \(\hat{\phi}\) component, so there will be no \(\hat{\phi}\) component of the bound volume current. The following is applicable in the region, \(0 \leq s \leq a\), the interior of the cylinder.

\[
\vec{J}_b = -\frac{\partial}{\partial z} \left( \frac{M_o s^2}{a^2} \right) \hat{s} + \frac{1}{s} \left[ \frac{\partial}{\partial s} \left( s \frac{M_o s^2}{a^2} \right) \right] \hat{z} \quad (23)
\]

\[
= \frac{1}{s} \left[ \frac{\partial}{\partial s} \left( s \frac{M_o s^3}{a^2} \right) \right] \hat{z} \quad (24)
\]

\[
= \frac{3 M_o s}{a^2} \hat{z} \quad (25)
\]
The next step is to show that these bound currents are not transferring charge. Notice that the bound surface current is directed opposite to the bound volume current (regardless of the sign of $M_o$). Surface charge density is given in units of Amperes per meter. The total current flowing along the cylinder’s surface is given by multiplying the magnitude of $\vec{K}_b$ by the length across which it flows, the circumference of the cylinder.

$$I_{surf} = |K| \cdot 2\pi a$$  \hspace{1cm} (26)$$

$$= 2\pi M_o a$$  \hspace{1cm} (27)$$

where I have not included the negative sign because it has been established that this current is in the direction opposite the bound volume current.

The total bound volume current is,

$$I_{vol} = \int \vec{J}_b \cdot d\vec{A}$$  \hspace{1cm} (28)$$

where $d\vec{A}$ is a differential surface area (the circular cross-section of the cylinder).

$$I_{vol} = \int_0^a \int_0^{2\pi} \frac{3M_o s}{a^2} \hat{z} \cdot s \ ds \ d\phi \ \hat{z}$$  \hspace{1cm} (29)$$

$$= \frac{3M_o}{a^2} (2\pi) \int_0^a s^2 \ ds$$  \hspace{1cm} (30)$$

$$= \frac{6\pi M_o}{a^2} \left[ \frac{s^3}{3} \right]_0^a$$  \hspace{1cm} (31)$$

$$= \frac{2\pi M_o}{a^2} a^3$$  \hspace{1cm} (32)$$

$$= 2\pi M_o a$$  \hspace{1cm} (33)$$

The currents in (27) and (33) are equal. Since they are in opposite directions this means that there is no net transfer of charge.

(a) Calculate $\vec{B}$ using the bound currents.

For $s > a$ we can immediately say that $\vec{B} = 0$. While determining that the bound currents result in no net charge transfer we have also shown that there is no current enclosed by an Amperian loop drawn around the cylinder.

For $s \leq a$, draw an Amperian loop inside the cylinder. This loop is centered on the cylinder’s axis and is directed along $\hat{\phi}$. Use (8) to solve for $\vec{B}$,
\[ B(2\pi s) = \mu_o \int \vec{J}_b \cdot d\vec{A} \quad (34) \]

\[ = \mu_o \int_0^s \int_0^{2\pi} \frac{3M_o s}{a^2} \hat{z} \cdot s \, ds \, d\phi \hat{z} \quad (35) \]

\[ = \mu_o \frac{3M_o}{a^2} (2\pi) \left( \frac{s^3}{3} \right) \quad (36) \]

\[ \vec{B} = \frac{\mu_o M_o s^2}{a^2} \hat{\phi} \quad (37) \]

where the direction of \( \vec{B} \) is known because the Amperian loop is in the same direction.

(b) Calculate \( \vec{B} \) using \( \vec{H} \) and the modified form of Ampere’s law.

We will be using the following equations to solve for the magnetic field by way of the auxiliary (\( \vec{H} \)) field,

\[ \vec{H} \equiv \frac{1}{\mu_o} \vec{B} - \vec{M} \quad \text{Eq. 6.18} \]

\[ \nabla \times \vec{H} = \vec{J}_f \quad \text{Eq. 6.19} \]

\[ \oint \vec{H} \cdot d\vec{l} = I_{f,enc} \quad \text{Eq. 6.20, integral form of 6.19} \quad (40) \]

The problem statement does not mention the existence of any free currents so \( I_{f,enc} = 0 \). Free currents are put into a system by us (e.g. plugging a circuit into a power supply or connecting a battery across a resistor), they do not spontaneously appear.

Using (40) and replacing \( \vec{H} \) by way of (38) we may apply Ampere’s law. In this case the closed loop is centered on the cylinder’s axis and is a circle of radius \( s \). The symmetry of the problem indicates that any magnetic field will be directed along \( \hat{\phi} \) (noting that \( d\vec{l} = s \, d\phi \hat{\phi} \)).

\[ \oint \left[ \frac{1}{\mu_o} \vec{B} - \vec{M} \right] \cdot d\vec{l} = 0 \quad (41) \]

\[ \oint \frac{1}{\mu_o} \vec{B} \cdot d\vec{l} = \oint \vec{M} \cdot d\vec{l} \quad (42) \]

\[ B(2\pi s) = \mu_o M(2\pi s) \quad (43) \]

\[ B = \mu_o M \quad (44) \]

where \( M \) is the magnitude of \( \vec{M} \).

The expression in (44) is general, but the value of \( M \) depends on which region we are examining. Outside of the cylinder, \( s > a \), there is no magnetization and the magnetic field is given by,

\[ \vec{B} = 0 \quad (45) \]
which agrees with the solution in (a). In part (a) we solved for the magnetic field using
conceptual arguments and our understanding of such fields. Using the modified form of
Ampere’s law did not provide a significant benefit in this case.

Inside the cylinder, \( s \leq a \), the magnetization is non-zero,

\[
B = \mu_o \left( \frac{M_o s^2}{a^2} \right)
\]

(46)

\[
\vec{B} = \frac{\mu_o M_o s^2}{a^2} \hat{\phi}
\]

(47)

which agrees with the solution in part (a). This time, using the modified form of Ampere’s
law was a considerably simpler method.

[4.] Problem 6.15 from Griffiths

Find the field inside the uniformly magnetized sphere of example 6.1 (p. 264). Take the
following as a hint: There is no free current in this sphere, therefore, the \( \vec{H} \) field may be
written as, \( \vec{H} = -\vec{\nabla} W \) where \( W \) is a scalar potential. By way of equation 6.23 it is also
known that, \( \nabla^2 W = \nabla \cdot \vec{M} \). Finally, \( \nabla \cdot \vec{M} = 0 \) everywhere except at the surface of the
sphere.

Solution

This is a problem concerning Laplace’s equation, \( \nabla^2 W = 0 \). It is known that the solutions to
Laplace’s equation in spherical geometry with azimuthal symmetry (i.e. no \( \phi \) dependence)
are of the form,

\[
\Xi = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)
\]

(48)

where \( \Xi \) is any function satisfying \( \nabla^2 \Xi = 0 \), the \( A \)'s and \( B \)'s are arbitrary constants, and
the \( P_l \) terms are the Legendre polynomials.

Inside the sphere there cannot be a \( 1/r \) term because that would represent an infinite \( W \) at
\( r = 0 \). Outside the sphere there can be no \( r \) term because that would represent an infinite \( W \)
at \( r \to \infty \). Therefore the expressions for \( W \) inside and outside the sphere may be partially
simplified and written as,

\[
W_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)
\]

(49)

\[
W_{\text{out}} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)
\]

(50)

This scalar potential is a continuous function (treat it like electric potential, \( V \), in the bound-
ary value problems of electrostatics), therefore we have our first boundary condition at the
surface of the sphere,

\[
W_{\text{in}}(r = R) = W_{\text{out}}(r = R)
\]

(51)

\[
\sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)
\]

(52)
The orthogonality of the Legendre polynomials means that we can equate each \( l \) term separately.

\[
A_l R^l = \frac{B_l}{R^{l+1}} \quad (53)
\]

The problem only asks for the field inside the sphere, but the field inside must still adhere to boundary conditions that relate to the field outside. In (53) we relate the \( A_l \) terms (which we want to solve) to the \( B_l \) terms that we do not care about. After finding another relation between these variables we can neglect the \( B_l \) terms and finish the problem.

There is another condition at the sphere’s surface that will provide the second equation we need to solve for the \( A_l \) terms.

\[
\mathbf{H}_\text{above} - \mathbf{H}_\text{below} = - (M^\perp_\text{above} - M^\perp_\text{below}) \quad \text{Eq. 6.24} \quad (54)
\]

where the \( \text{above} \) and \( \text{below} \) subscripts are equivalent to \( \text{outside} \) and \( \text{inside} \) respectively.

In this spherical geometry the perpendicular components at \( r = R \) are those along the \( \hat{r} \) direction. Beginning with the right side of (54), \( M_\text{above} = M_\text{out} = 0 \) since there is no magnetization outside of the sphere. This means that the perpendicular component of \( \mathbf{M} \) must also be zero.

From example 6.1 we may write the magnetization of the sphere as \( \mathbf{M} = M_o \hat{z} \), where \( M_o \) is a constant. Using \( \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \), the \( \hat{r} \) component of \( \mathbf{M}_\text{in} \) is,

\[
M^\perp_{\text{in}} = M_o \cos \theta \quad (55)
\]

and this is the right side of (54).

For the left side of (54) we keep only the \( \hat{r} \) component of the \( \mathbf{H} = -\nabla W \) expression. Taking the expression for the gradient in spherical coordinates we get (recalling that there is already a negative sign in the expression),

\[
\begin{align*}
\mathbf{H}_\text{out}^\perp &= -\frac{\partial}{\partial r} W_\text{out} = -\sum_{l} \frac{(l+1)B_l}{r^{l+2}} P_l(\cos \theta) = \sum_{l} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos \theta) \quad (56) \\
\mathbf{H}_\text{in}^\perp &= -\frac{\partial}{\partial r} W_\text{in} = -\sum_{l} l A_l r^{l-1} P_l(\cos \theta) = -\sum_{l} l A_l R^{l-1} P_l(\cos \theta) \quad (57)
\end{align*}
\]

where the last step in each line incorporated the fact that \( r = R \).

Taking all of the information just calculated allows us to rewrite (54) as,

\[
\sum_{l} \frac{(l+1)B_l}{R^{l+2}} P_l(\cos \theta) - \left( \sum_{l} l A_l R^{l-1} P_l(\cos \theta) \right) = M_o \cos \theta \quad (58)
\]

The right side includes a \( \cos \theta \) term, which is equivalent to \( P_1(\cos \theta) \). As such, the left side of the above expression must have only \( l = 1 \) terms. All other values of \( A_l \) and \( B_l \) are zero. Rewriting (58),

\[
\frac{(2)B_1}{R^3} P_1(\cos \theta) + A_1 R^0 P_1(\cos \theta) = M_o P_1(\cos \theta) \quad (59)
\]

\[
\frac{2B_1}{R^3} + A_1 = M_o \quad (60)
\]
From (53) we have,

\[ B_1 = R^3 A_1 \]  

(61)

and this may be put into (60) as follows,

\[ \frac{2}{R^3} R^3 A_1 + A_1 = M_o \]  

(62)

\[ A_1 = \frac{M_o}{3} \]  

(63)

The scalar potential inside the sphere is,

\[ W_{in} = A_1 r \cos \theta = \frac{M_o}{3} r \cos \theta \]  

(64)

Now we can solve for \( \vec{H} \) inside the sphere,

\[ \vec{H}_{in} = -\nabla W_{in} = - \left[ \frac{\partial}{\partial r} \frac{M_o}{3} r \cos \theta \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \frac{M_o}{3} r \cos \theta \hat{\theta} \right] \]  

(65)

\[ = -\frac{M_o}{3} \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right) \]  

(66)

\[ = -\frac{M_o}{3} \hat{z} \]  

(67)

The magnetic field, \( \vec{B} \), may be found using (38) and the fact that \( \vec{M}_{in} = M_o \hat{z} \),

\[ \vec{B}_{in} = \mu_o (\vec{H}_{in} + \vec{M}_{in}) \]  

(68)

\[ = \mu_o \left( -\frac{M_o}{3} \hat{z} + M_o \hat{z} \right) \]  

(69)

\[ = \frac{2}{3} \mu_o M_o \hat{z} \]  

(70)

\[ = \frac{2}{3} \mu_o \vec{M} \]  

(71)

where (71) is the solution given as Griffiths equation 6.16. The method used here is just another way to solve the problem.

[5.] **Professor Carter Problem**

A coaxial cable consists of a conducting wire of radius \( a \) surrounded by a conducting cylindrical tube of radius \( c \). A current \( I \) flows down the wire and returns along the outer conductor (see figure 6.24), in both cases the current is distributed uniformly. Two different magnetic materials exist in the space between the two conductors of the cable: the first, with susceptibility \( \chi_{m,1} \), fills the space from \( a < r < b \) and the second, with susceptibility \( \chi_{m,2} \), fills the space from \( b < r < c \).

(a) Find the magnetic field in the region between the conductors.
(b) Calculate the magnetization and find the bound currents; make sure these are consistent with the answer to part (a).

**Solution**

Begin by noting that the problem writes the radial coordinate in terms of $r$, while Griffiths uses $s$. This solution will use $s$ so that the method better matches that given in the text and used on other problems.

(a) Let the current in the center wire be along the $\hat{z}$ direction. The current in the outer shell must then be along the $-\hat{z}$ direction, but this will not matter since we only need to solve for the fields inside the coaxial cable. The free currents are given in the problem and they exhibit cylindrical symmetry. We can use (40) to solve for $\vec{H}$ and then find $\vec{B}$.

\[
H_\phi(2\pi s) = I_f \tag{72}
\]

\[
\vec{H}_m = \frac{I}{2\pi s} \hat{\phi} \tag{73}
\]

where $\vec{H}_m$ is the field in the region between the conductors. This does not depend on the materials inside the coaxial cable because $\vec{H}$ is determined entirely from free currents.

There is a relation between $\vec{B}$ and $\vec{H}$ fields in linear media (it is safe to assume that all magnetic materials studied in this course will be linear),

\[
\vec{B} = \mu_0 (1 + \chi_m) \vec{H} \quad \text{Eqs. 6.31 and 6.32} \tag{74}
\]

This shows how the materials affect the field. There is a value of $\vec{B}$ for the region occupied by each material. Calling these regions 1 and 2 for containing materials with $\chi_{m,1}$ and $\chi_{m,2}$ respectively,

\[
\vec{B}_1 = \mu_0 (1 + \chi_{m,1}) \frac{I}{2\pi s} \hat{\phi} \tag{75}
\]

\[
\vec{B}_2 = \mu_0 (1 + \chi_{m,2}) \frac{I}{2\pi s} \hat{\phi} \tag{76}
\]

(b) In linear media the magnetization is given by,

\[
\vec{M} = \chi_m \vec{H} \quad \text{Eq. 6.29} \tag{77}
\]

The magnetization will therefore vary according to which material we are considering. This is similar to the manner in which the polarization in a region of space depends on the dielectric properties of any materials there. The magnetization in each region is,

\[
\vec{M}_1 = \chi_{m,1} \frac{I}{2\pi s} \hat{\phi} \tag{78}
\]

\[
\vec{M}_2 = \chi_{m,2} \frac{I}{2\pi s} \hat{\phi} \tag{79}
\]

where $\vec{H}$ from part (a) has been used (it’s still valid because nothing in the problem has changed, we are just applying a different method to solve for the magnetic field).
Once again, the bound current densities are given by (1). Beginning with the volume bound current densities,

\[ \vec{J}_{b,1} = \nabla \times \frac{\chi_{m,1}I}{2\pi s} \hat{\phi} = \frac{1}{s} \left[ \frac{\partial}{\partial s} \left( s \cdot \frac{\chi_{m,1}I}{2\pi s} \right) \right] \hat{z} = 0 \] (80)

\[ \vec{J}_{b,2} = \nabla \times \frac{\chi_{m,2}I}{2\pi s} \hat{\phi} = \frac{1}{s} \left[ \frac{\partial}{\partial s} \left( s \cdot \frac{\chi_{m,2}I}{2\pi s} \right) \right] \hat{z} = 0 \] (81)

There is no bound volume current.

There are three surfaces at which to determine bound surface current density, \( s = a, s = b, \) and \( s = c \). This is where the surfaces of the magnetic materials may be found. For \( s = a \) only the magnetization of material 1 matters, and the normal vector to the surface is directed along \( -\hat{s} \),

\[ \vec{K}_b = \frac{\chi_{m,1}I}{2\pi a} \hat{\phi} \times (-\hat{s}) = \frac{\chi_{m,1}I}{2\pi a} \hat{z} \quad \text{at } s = a \] (82)

At \( s = b \) there are two separate bound surface current densities to consider. Each material makes a contribution and the total density on that surface is their sum. The direction of the normal vector is reversed for each region.

\[ \vec{K}_{b,1} = \frac{\chi_{m,1}I}{2\pi b} \hat{\phi} \times \hat{s} = -\frac{\chi_{m,1}I}{2\pi b} \hat{z} \] (83)

\[ \vec{K}_{b,2} = \frac{\chi_{m,2}I}{2\pi b} \hat{\phi} \times (-\hat{s}) = \frac{\chi_{m,2}I}{2\pi b} \hat{z} \] (84)

The total surface current density at \( s = b \) is,

\[ \vec{K}_b = \frac{I}{2\pi b} (\chi_{m,2} - \chi_{m,1}) \hat{z} \quad \text{at } s = b \] (85)

At \( s = c \) the same method is employed,

\[ \vec{K}_b = \frac{\chi_{m,2}I}{2\pi c} \hat{\phi} \times \hat{s} = -\frac{\chi_{m,2}I}{2\pi c} \hat{z} \quad \text{at } s = c \] (86)

All of the bound currents are directed along the \( \hat{z} \) axis (positive or negative). Ampere’s law in the form of (8) can be used to find \( \vec{B} \) and make sure it agrees with the value given in part (a). Cylindrical symmetry is maintained, so we have,

\[ B_\phi(2\pi s) = \mu_0 I_{\text{enc}} \] (87)

where \( I_{\text{enc}} \) is the total enclosed current (bound and free). The free current is a simple constant that is given. The bound current is found from the bound surface current density. This density is constant, so the total bound current on any surface is found by multiplying the current density by the length across the surface. This length is the circumference in our cylindrically symmetric system.
In region 1 our Amperian loop encloses the free current and the current on the surface at 

\[ s = a, \]

\[ I_{\text{enc}} = I + K_{b,a}L = I + \frac{\chi_{m,1}I}{2\pi a}(2\pi a) = I + \chi_{m,1}I = (1 + \chi_{m,1})I \quad (88) \]

The magnetic field in region 1 is,

\[ \vec{B}_1 = \mu_0(1 + \chi_{m,1}) \frac{I}{2\pi s} \hat{\phi} \quad (89) \]

which agrees with the solution in part (a).

Region 2 has a different value for the total enclosed current,

\[ I_{\text{enc}} = I_f + I_{s=a} + I_{s=b} = I + \chi_{m,1}I + \left[ (\chi_{m,2} - \chi_{m,1}) \frac{I}{2\pi b} \right] \cdot 2\pi b \quad (90) \]

\[ = I + \chi_{m,1}I + \chi_{m,2}I - \chi_{m,1}I = (1 + \chi_{m,2})I \quad (91) \]

The magnetic field in region 2 is,

\[ \mu_0(1 + \chi_{m,2}) \frac{I}{2\pi s} \hat{\phi} \quad (92) \]

which agrees with the solution in part (a).

In this problem the magnetic materials can increase or decrease the field. In linear materials they do not change the direction of the field (e.g. a field originally directed along \( \hat{\phi} \) will not be redirected along \( \hat{z} \) in the presence of a linear magnetic material). Taking this into account it is sensible that the magnetic field inside the cable retains the \( 1/s \) dependence that we normally see in the field due to a current along a wire.

[6.] Professor Carter Problem

A conducting slab, parallel to the \( xy \) plane and extending from \( z = -a \) to \( z = a \), carries a uniform free current density \( \vec{j} = j_o \hat{x} \). The conductor (with \( \chi_m = 0 \)) is surrounded by a material with susceptibility \( \chi_m = 0 \) (you can imagine that this material covers all of space). Find the magnetic field (everywhere in space) and the location of all bound currents. (Do the bound currents transfer net charge in this case?)

Solution

Figure 4 shows the set up for this problem. This is a modified form of problem 5.14 in Griffiths (see figure 5.41 in Griffiths for another reference).

The slab is a conductor, therefore the magnetic field inside of it may be determined using the “unmodified” version of Ampere’s law given in (8). Figure 5 shows the Amperian loop used to determine this field (we will extend this loop to solve for the field outside of the conductor).

The magnetic field inside the conductor has no \( z \) component. Each individual current element generates a field whose \( z \) component cancels out the \( z \) component of the field generated by the neighboring current element. The resulting field is directed entirely along
Figure 4: Geometry of the problem. The slab is infinite in the $xy$ plane and has a thickness $\Delta z = 2a$. The placement of the coordinate axes is arbitrary, so long as they correctly represent an orthogonal triplet (i.e. $\hat{x} \times \hat{y} = \hat{z}$)

Figure 5: A view of the conducting slab in the $yz$ plane. The horizontal axis is the $y$ direction, and the vertical axis is the $z$ direction. The current is coming out of the plane of the page, as shown by the vector dot. The magnetic field due to one small element of the current is shown. All of the individual current elements add together to generate a $\vec{B}$ that is directed entirely along the $\pm \hat{y}$ direction.

$\pm \hat{y}$. In the following steps I am going to ignore direction since we have solved for that based on the geometry of the problem. Using the dashed line in figure 5 as our Amperian loop leads to,

$$B(2y) = \mu_o(2yz)j_o \tag{93}$$

$$B = \mu_o j_o \hat{z} \tag{94}$$

$$\vec{B} = -\mu_o j_o \hat{z} \hat{y} \tag{95}$$
where in (95) I have included the vector properties of the magnetic field. In this case, the negative sign is included because we know that in the $+z$ ($-z$) plane the magnetic field is directed along $-\hat{y}$ ($+\hat{y}$).

To solve for the magnetic field outside of the conductor we can stretch the previously used Amperian loop along the $z$ direction. The modified form of Ampere’s law, (40), can now be used. Equivalent to the previous case, the $\vec{H}$ field will have no $z$ component because of the way contributions to the field from neighboring current elements cancel out. This simplification stems from this conductor being of infinite extent in the $xy$ plane. One difference in this case is that the total free current enclosed by our loop will have no $z$ dependence because we are only concerned with the region $z > |a|$.

\[
H(2y) = (2ay)j_o \tag{96}
\]

\[
H = aj_o \tag{97}
\]

\[
\vec{H} = \begin{cases} 
-aj_o \hat{y} & \text{for } z > a \\
aj_o \hat{y} & \text{for } z < -a
\end{cases} \tag{98}
\]

where in (98) I have assigned the directions of the field. This $\vec{H}$ field has no $z$ dependence so it must be written in a form like that above. Use (74) to solve for the magnetic field outside of the conductor by way of our $\vec{H}$ field,

\[
\vec{B} = \mu_o(1 + \chi_m)\vec{H} \tag{99}
\]

where $\vec{H}$ is given in (98).

The bound currents may be found once we know the magnetization, $\vec{M}$. Magnetization is related to the $\vec{H}$ field according to (77), $\vec{M} = \chi_m\vec{H}$. The expressions for the bound currents are given in (1). There are two surfaces of interest: the $xy$ planes at $z = a$ ($\hat{n} = -\hat{z}$) and at $z = -a$ ($\hat{n} = +\hat{z}$). Recall that the direction of the surface normal is determined by pointing away from the material being investigated.

\[
\vec{K}_{b,z=a} = \chi_m(-aj_o) \hat{y} \times (-\hat{z}) = \chi_maj_o \hat{x} \tag{100}
\]

\[
\vec{K}_{b,z=-a} = \chi_m(aj_o) \hat{y} \times \hat{z} = \chi_maj_o \hat{x} \tag{102}
\]

Since $\vec{H}$ is uniform, that means $\vec{\nabla} \times \vec{H} = 0$ and there are no bound volume currents, $\vec{J}_b = 0$.

This solution implies that there is a net transfer of charge due to bound currents. The appearance of a charge transfer is due to the fact that this is not a real, physical system. The surface bound currents are going in the same direction, but they must be turning around at infinity and eventually closing in on themselves. We cannot solve for these closed current paths (one above the conductor and one below) because the total surface current is infinite. Put another way, ask yourself how much current is flowing along the $z = +a$ surface? This
is given by, $I = |\vec{K}| \cdot l$, where $l$ is the length of the line across which the surface current flows. The surface extends to infinity, so $I = \infty$. This is a non-physical result. As such, our previous requirements for bound currents are modified and we see an apparent charge transfer. Rest assured, in any real, physical system this will not be the case.

[7.] Problem 6.18 from Griffiths

A sphere is placed in a uniform background magnetic field. The sphere is made of linear magnetic material and the background field is $\vec{B}_o$. Find the magnetic field inside the sphere. (Prof. Carter Hint: use the method of problem 6.15, referring to example 4.7)

Solution

Professor Carter solved this problem using an iterative method in lecture (October 5, 2005). Here we will follow the hint and solve this as a separation of variables problem.

There is no free current in the sphere; we know this because free current would only exist in the sphere if it was purposely put there, and then the problem statement would describe it. Without free current the curl of $\vec{H}$ is zero and have (just as problem 6.15 describes),

$$\vec{H} = -\vec{\nabla}W$$  \hspace{1cm} (104)

$$\nabla^2 W = 0 \quad \text{Inside and outside the sphere}$$  \hspace{1cm} (105)

where $W$ is again a scalar potential. The plan is to solve for $W$ inside the sphere, use that value to solve for $\vec{H}_{in}$, and then use $\vec{H}_{in}$ to solve for $\vec{B}_{in}$. Inside the sphere we have,

$$W_{in} = \sum_{l}^{\infty} A_l r^l P_l(\cos \theta)$$  \hspace{1cm} (106)

which is the given solution for Laplace’s equation in spherically symmetric systems. Before we can write the expression for $W$ outside the sphere we have to determine the value of $W$ as $r \to \infty$. The background field is $\vec{B}_o = B_o \hat{z}$. The scalar potential is in terms of $\vec{H}$ and leads to (see example 4.7 in Griffiths to prove this to yourself),

$$W(r \to \infty) = -H_o r \cos \theta = -\frac{B_o}{\mu_o} r \cos \theta$$  \hspace{1cm} (107)

where this makes use of $B = \mu_o H$ outside the sphere. The expression for the potential outside of the sphere contains this term and the general sum,

$$W_{out} = -\frac{B_o}{\mu_o} r \cos \theta + \sum_{l}^{\infty} \frac{C_l}{r^{l+1}} P_l(\cos \theta)$$  \hspace{1cm} (108)

where $C_l$ is an unknown constant.

At the surface of the sphere, $W_{in}(r = R) = W_{out}(r = R)$,

$$\sum_{l}^{\infty} A_l R^l P_l(\cos \theta) = -\frac{B_o}{\mu_o} R \cos \theta + \sum_{l}^{\infty} \frac{C_l}{R^{l+1}} P_l(\cos \theta)$$  \hspace{1cm} (109)
Noting that \( \cos \theta = P_1(\cos \theta) \), (109) gives two separate expressions,

\[
A_1 R = -\frac{B_o}{\mu_o} R + \frac{C_1}{R^2} \quad \text{for } l = 1
\]

\[
A_l R^l = \frac{C_1}{R^{l+1}} \quad \text{for } l \neq 1
\]

In the equations above there are (essentially) two unknowns and one equation. We can use another boundary condition at the surface of the sphere to get another equation. In problem 6.15 we used (54), but in this problem we don’t know anything about the magnetization inside the sphere. We do, however, have information about the magnetic field across this boundary and we may consider,

\[
B_{\text{above}} - B_{\text{below}} = 0 \quad \text{Eq. 6.26} (112)
\]

Solving for \( B_{\text{above}} \), which is evaluated at \( r = R \),

\[
B_{\text{above}} = \mu_o H_{\text{out}} = -\mu_o \frac{\partial}{\partial r} W_{\text{out}}
\]

\[
= -\mu_o \left[ -\frac{B_o}{\mu_o} \cos \theta + \sum_{l} \frac{-(l + 1)C_l}{R^{l+2}} P_l(\cos \theta) \right] \quad (114)
\]

\[
= B_o P_1(\cos \theta) + \sum_{l} \frac{\mu_o(l + 1)C_l}{R^{l+2}} P_l(\cos \theta) \quad (115)
\]

where the perpendicular components are the \( \hat{r} \) components of the gradient.

To find \( B_{\text{below}} \) we follow a similar method, but in this case \( B = \mu H \), where \( \mu = \mu_o(1 + \chi_m) \). Since the sphere is made of linear magnetic material we know that its susceptibility may be written as \( \chi_m \).

\[
B_{\text{below}} = \mu H_{\text{in}} = -\mu \frac{\partial}{\partial r} W_{\text{in}}
\]

\[
= -\sum_{l} \mu_l A_l R^{l-1} P_l(\cos \theta) \quad (117)
\]

The expressions (117) and (115) are equal according to the boundary condition given in (112).

\[
-B_o - \frac{2\mu_o C_1}{R^3} = \mu_o(1 + \chi_m) A_1 \quad \text{for } l = 1
\]

\[
-\frac{(l + 1)C_l}{R^{l+2}} = lA_l R^{l-1} \quad \text{for } l \neq 1
\]

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Solving for the \( l \neq 1 \) terms first, insert (111) into (119),

\[-\frac{(l + 1)}{R^{l+2}} (A_l R^{2l+1}) = l A_l R^{l-1} \]  \hspace{1cm} (120)
\[-(l + 1) R^{2l+1} = l R^{l-1} R^{l+2} \]  \hspace{1cm} (121)
\[-(l + 1) R^{2l+1} = l R^{2l+1} \]  \hspace{1cm} (122)
\[-l - 1 = l \]  \hspace{1cm} (123)
\[l = -\frac{1}{2} \]  \hspace{1cm} (124)

so the \( l \neq 1 \) terms only exist for the \( l = -1/2 \) value. Our sum for the potential requires that the \( l \)'s be positive whole numbers, therefore, the only terms that exist in the final solution are the \( l = 1 \) terms.

Solve for the \( l = 1 \) terms by inserting (118) into (110),

\[A_1 R = -\frac{B_o}{\mu_o} R + \frac{1}{R^2} \left[ -\frac{R^3}{2\mu_o} (\mu_o (1 + \chi_m) A_1 + B_o) \right] \]  \hspace{1cm} (125)
\[A_1 = -\frac{B_o}{\mu_o} - \frac{1}{2\mu_o} (\mu_o (1 + \chi_m) A_1 + B_o) \]  \hspace{1cm} (126)
\[A_1 \left( 1 + \frac{\mu_o (1 + \chi_m)}{2\mu_o} \right) = -\frac{3B_o}{2\mu_o} \]  \hspace{1cm} (127)
\[A_1 = -\frac{3B_o}{2\mu_o} \left( \frac{2\mu_o}{2\mu_o + \mu_o (1 + \chi_m)} \right) \]  \hspace{1cm} (128)
\[A_1 = -\frac{3B_o}{\mu_o} \left( \frac{1}{3 + \chi_m} \right) \]  \hspace{1cm} (129)

For the sake of eventually matching the solution presented in lecture, I will rewrite \( A_1 \) as,

\[A_1 = -\frac{B_o}{\mu_o} \left( \frac{1}{1 + \frac{\chi_m}{3}} \right) \]  \hspace{1cm} (130)

Now that we have \( A_1 \) we know the value of \( W \) inside the sphere,

\[W_{\text{in}} = -\frac{B_o}{\mu_o} \left( \frac{1}{1 + \frac{\chi_m}{3}} \right) r \cos \theta \]  \hspace{1cm} (131)
The value of $\vec{H}_{in}$ is,

$$
\vec{H}_{in} = -\nabla W_{in} = - \left[ \frac{\partial}{\partial r} W_{in\hat{r}} + \frac{1}{r} \frac{\partial}{\partial \theta} W_{in\hat{\theta}} \right]
$$

(132)

$$
= - \left[ - \frac{B_o}{\mu_o} \left( \frac{1}{1 + \frac{\chi_m}{3}} \right) \right] \left( \cos \theta \hat{r} + \frac{1}{r} ( - \sin \theta ) \hat{\theta} \right)
$$

(133)

$$
= \frac{B_o}{\mu_o} \left( \frac{1}{1 + \frac{\chi_m}{3}} \right) \hat{z}
$$

(134)

Finally, the magnetic field inside the sphere is,

$$
\vec{B}_{in} = \mu \vec{H}_{in} = \mu_o (1 + \chi_m) \cdot \frac{B_o}{\mu_o} \left( \frac{1}{1 + \frac{\chi_m}{3}} \right) \hat{z}
$$

(135)

$$
= \frac{1 + \chi_m}{1 + \frac{\chi_m}{3}} B_o \hat{z}
$$

(136)

which agrees with the iterative method of lecture.